A NOTE ON WEIGHTED SOBOLEV SPACES,
AND REGULARITY OF COMMUTATORS
AND LAYER POTENTIALS ASSOCIATED TO THE HEAT EQUATION

STEVE HOFMANN

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Abstract. We give a simplified proof of recent regularity results of Lewis and Murray, namely, that certain commutators, and the boundary single layer potential for the heat equation in domains in $\mathbb{R}^2$ with time dependent boundary, map $L^p$ into an appropriate homogeneous Sobolev space. The simplification is achieved by treating directly only the case $p = 2$, but in a weighted setting.

1. Introduction and statement of results

Regularity results for certain commutators and layer potentials associated to the heat equation in domains in $\mathbb{R}^2$ with time dependent boundary have recently been obtained by Lewis and Murray [LM]. They showed that these operators are bounded from $L^p$ into an appropriate Sobolev space $I_\alpha(L^p)$. Their proof proceeds in two steps: First treat the case $p = 2$, and then use a variety of real variable techniques to extend to the case $p \neq 2$. Their results are equivalent to the $L^p$ boundedness of certain “nonstandard” singular integrals that, in particular, need not map constants into $\text{BMO}$; thus the $T_1$ Theorem does not apply, nor can one interpolate with an end point estimate to obtain the case $2 < p < \infty$. Not surprisingly then, the second part of their program entails a not inconsiderable expenditure of effort, and it would therefore seem desirable to dispense with this step entirely. Fortunately, there is a way to do this: in the (possibly apocryphal) words of Rubio de Francia, “$L^p$ does not exist, only (weighted)$L^2$.” In this note we will prove a weighted version of the $L^2$ result of [LM], from which most of the $L^p$ theory follows automatically (in particular, we obtain the case of principal interest in parabolic theory, namely, $\alpha = \frac{1}{2}$ for all $p$, $1 < p < \infty$). While the weighted results are new and perhaps of independent interest, our primary motivation in establishing them is to simplify the arguments of [LM].

Before stating our theorems, we need to recall some elementary facts about Littlewood-Paley theory in $\mathbb{R}^n$. Let $\psi \in C^\infty_0(\mathbb{R}^n)$ be radial, be supported in the unit ball, and have mean value zero. We define $Q_s f \equiv \psi_s * f$, where $\psi_s(x) \equiv$
where $\psi$ has been normalized so that $\int_0^\infty (\psi(s\xi))^2 \, ds/s = 1$ for all $\xi \in \mathbb{R}^n$ (this can be done since $\psi$ is radial). Thus $Q_s$ satisfies the "Calderon reproducing formula"

$$\int_0^\infty Q_s^2 \, ds/s = I.$$  

For $0 < \alpha < 1$, we define $\tilde{Q}_s \equiv s^{-\alpha} I_\alpha Q_s$, where as usual $I_\alpha$ denotes the fractional integral operator

$$I_\alpha f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{f(x+t) - f(x)}{t^{1+\alpha}} \, dt.$$  

Then, at least for test functions,

$$\int_0^\infty \tilde{Q}_s^2 \, ds/s = C_\alpha I$$

since

$$0 < \int_0^\infty s^{-2\alpha}(\psi(s))^2 \, ds/s = C_\alpha < \infty, \quad 0 < \alpha < 1.$$  

The latter inequality follows from the smoothness of $\psi$ and the fact that $\psi(0) = 0$ (so, in particular, $|\psi(s)/s|$ is bounded near the origin). If we set

$$\hat{\psi}(|\xi|) \equiv |\xi|^{-\alpha} \hat{\psi}(|\xi|)$$

then a routine computation shows that

$$|\tilde{\psi}_s(x)| \leq \frac{C_{n,\alpha}s^{1-\alpha}}{(s + |x|)^{n+1-\alpha}}$$

and

$$|\nabla \tilde{\psi}_s(x)| \leq \frac{C_{n,\alpha}s^{1-\alpha}}{(s + |x|)^{n+2-\alpha}}.$$  

Thus,

$$\left( \int_0^\infty |\tilde{\psi}_s(x)|^2 \, ds/s \right)^{1/2} \leq C|x|^{-n}$$

and

$$\left( \int_0^\infty |\nabla \tilde{\psi}_s(x)|^2 \, ds/s \right)^{1/2} \leq C|x|^{-n-1}.$$  

By vector-valued Calderon-Zygmund theory (see, e.g., [GR, Chapter V]), we then have, for all $w \in A_2$,

$$\|\tilde{g}(f)\|_{2,w} \approx \|f\|_{2,w}, \quad (1.1)$$

where

$$\tilde{g}(f) \equiv \left( \int_0^\infty |\tilde{Q}_s f|^2 \, ds/s \right)^{1/2}.$$  

Furthermore, the homogeneous weighted Sobolev space $I_\alpha(L^2_w)$, $w \in A_2$, can be given the norm

$$\|f\|_{I_\alpha(L^2_w)} \equiv \left( \int_{\mathbb{R}^n} \int_0^\infty |Q_s f(x)|^2 \, ds/s \, w(x) \, dx \right)^{1/2}, \quad (1.2)$$

because by (1.1) this last expression is comparable to $\|\tilde{f}\|_{2,w}$, where $f = I_\alpha(\tilde{f}).$  

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For $0 < \alpha < 1$, let (real-valued) $A \in I_\alpha(BMO)$, and consider the one-dimensional operator

\[
K_\alpha f(x) \equiv \int_{\mathbb{R}} \frac{[A(x) - A(y)]^2}{|x - y|^{1+\alpha}} f(y) \, dy.
\]

As in [LM], our results for this operator can easily be extended to the boundary single layer potential for the heat equation in domains $\{(x_1, x_2) : x_1 > A(x_2)\}$, with $A = I_{1/2a}$, $a \in BMO$. We shall return to this point in §4. Our principal result is the following.

**Theorem 1.4.** Let $A = I_\alpha a$, and suppose $w \in A_1$ if $0 < a < 1$ or $w \in A_2$ if $\frac{1}{2} \leq a < 1$. Then

\[
\|K_\alpha f\|_{L^p(A_1 w)} \leq C_\alpha \|a\|_w^2 \|f\|_{2,w}.
\]

As an almost immediate corollary, we recover, except for the case $1 < p < 2$, $0 < \alpha < \frac{1}{2}$, the result of [LM, Theorem 3].

**Theorem 1.5.** Suppose $1 < p < \infty$ if $\frac{1}{2} \leq \alpha < 1$, or $2 < p < \infty$ if $0 < \alpha < 1$. Then

\[
\|K_\alpha f\|_{L^p(A_1 w)} \leq C_{\alpha,p} \|a\|_w^2 \|f\|_p.
\]

**Proof of Theorem 1.5** (Modulo Theorem 1.4). If $\frac{1}{2} \leq \alpha < 1$, then by Theorem 1.4 we have that $D\alpha K_\alpha$ is bounded on $L^2_{w}, w \in A_2$, and therefore on $L^p_w$, $1 < p < \infty$, $w \in A_p$, by Rubio de Francia’s extrapolation theorem (see, e.g., [GR, Chapter IV]), where

\[
(D^\alpha f)(\xi) \equiv |\xi|^\alpha \hat{f}(\xi).
\]

If $0 < \alpha < 1$ then $D^\alpha K_\alpha$ is bounded on $L^2_w$, $w \in A_1$. In particular, by a result of Coifman and Rochberg [CR], we have for $u \in L^{(p/2)'}$, $p > 2$, the inequality

\[
\int |D^\alpha K_\alpha f(x)|^2 u(x) \, dx \leq C_{\alpha,\varepsilon} \|a\|_w^4 \int |f(x)|^2 (M(|u|^{1+\varepsilon}))^{1+\varepsilon}(x) \, dx,
\]

for any positive $\varepsilon$. By choosing $1 + \varepsilon < (p/2)'$, the $L^p$ boundedness, $p > 2$, of $D^\alpha K_\alpha$ may be deduced by a standard duality argument. Theorem 1.5 follows.

We remark that the proof to follow will actually show that in the case $0 < \alpha < \frac{1}{2}$, one may take $w \in A_{1+2\alpha}$ in Theorem 1.4, and therefore by a slightly more involved duality argument, one obtains Theorem 1.5 for $p > 2/(1 + 2\alpha)$. This is the best result that can be directly obtained by our method, which relies on the auxiliary use of the Littlewood-Paley $g^*_\lambda$ function, with $\lambda < 1 + 2\alpha$. Since the full range of $p$ has already been treated in [LM], we shall content ourselves with Theorems 1.4 and 1.5 as stated.

In the next section, we give a transparent extension to the weighted setting of a result of Strichartz relating Carleson measures and $I_\alpha(BMO)$. In §3 we prove Theorem 1.4, and then in §4 we describe how this result may be extended to the boundary single layer potential.
2. **$I_a(\text{BMO})$ and Weighted Carleson Measures**

We first need a preliminary fact.

**Lemma 2.1.** With $0 < \alpha < 1$, the square function $g_\alpha f$ defined on $\mathbb{R}^n$ by

$$g_\alpha f(x) \equiv \left( \int_{\mathbb{R}^n} |I_\alpha f(x + h) - I_\alpha f(x)|^2 \frac{dh}{|h|^{n+2\alpha}} \right)^{1/2}$$

is bounded on $L^2_w$, $w \in A_p(\alpha)$, where $p(\alpha) \equiv \min(1 + 2\alpha/n, 2)$. In particular, we may always take $w \in A_1$, and if $n = 1$ and $\frac{1}{2} \leq \alpha < 1$, we may take $w \in A_2$.

**Proof of Lemma 2.1.** This is fairly trivial. First (see, e.g., Stein [S, pp. 162–163, 6.12, 6.13] and the references given therein) we have the pointwise bound

$$g_\alpha f(x) \leq C_{\alpha,\lambda} g_\lambda^* f(x)$$

if $\lambda < 1 + 2\alpha/n$ (see [S, p. 88] for the definition of $g_\lambda^*$). But by a result of Muckenhoupt and Wheeden [MW], $g_\lambda^*$ is bounded on $L^2_w$, with $w \in A_q(\lambda)$, $q(\lambda) = \min(\lambda, 2)$, $\lambda > 1$. If $1 + 2\alpha/n > 2$, we take $\lambda = 2$. If $1 + 2\alpha/n < 2$ and $w \in A_1 + 2\alpha/n$, then by a well-known property of $A_p$ weights we may select a $\lambda < 1 + 2\alpha/n$, with $w \in A_2$. In either case, Lemma 2.1 follows by [MW] and the pointwise bound (2.2).

We now give a weighted version of Theorem (3.3) of [Stz].

**Lemma 2.3.** Suppose $0 < \alpha < 1$ and $A = I_\alpha a$, $a \in \text{BMO}$. If $Q(s)$ is a cube with side length $s$ and $p(\alpha) \equiv \min(1 + 2\alpha/n, 2)$, then

$$\int_{Q(s)} \int_{|h| \leq s} \frac{|A(x + h) - A(x)|^2}{|h|^{n+2\alpha}} dhw(x) dx \leq C_{\alpha} \|a\|_{L^2_w}$$

where $w \in A_p(\alpha)$. In particular, we may always take $w \in A_1$, and if $\frac{1}{2} \leq \alpha < 1$ and $n = 1$, we may take $w \in A_2$.

**Proof of Lemma 2.3.** This is easy if we follow the argument in [Stz], combined with that of Journe [J, pp. 85–87], so we only give a brief sketch. Since the operator

$$f \rightarrow I_\alpha f(x + h) - I_\alpha f(x)$$

has Fourier multiplier $[e^{ih\xi} - 1] |\xi|^{\alpha}$, it annihilates constants, so we may assume that $a$ has mean value zero on $Q^*(s)$. Here $Q^*(s)$ denotes the cube concentric with $Q(s)$ and has ten times the diameter of $Q(s)$. As usual, we write $a = a_1 + a_2$, where $a_1 \equiv a_1 Q^*(s)$, $a_2 = a_2(x Q^*(s)) c$. Now crudely, by Lemma 2.1, the left side of (2.4) with $I_\alpha a_1$ in place of $A$ is no larger than a constant times

$$\frac{1}{w(Q(s))} \int |a_1(x)|^2 w(x) dx.$$

The desired estimate for this last term may be obtained exactly like the corresponding estimate in [J, p. 86] by using Hölder’s inequality, the reverse Hölder property of $A_p$ weights, the John-Nirenberg Theorem, and the fact that $w$ defines a doubling measure. To handle the part of (2.4) corresponding to $I_\alpha a_2$,
we observe first that the operator defined by (2.5) is given by convolution with the kernel
\[ C_n \left[ \frac{1}{|x+h|^{n-\alpha}} - \frac{1}{|x|^{n-\alpha}} \right] \leq C_{n, \alpha} \frac{|h|}{|x|^{n+1-\alpha}}, \]
where the last inequality holds whenever $|x| > 2|h|$. If we write $h = t\theta$ in polar coordinates, then the left side of (2.4) with $I_{\alpha}a_2$ in place of $A$ is bounded by
\[ \int_{s^{-1}} \int_{Q(s)} \int_{0}^{5} \left[ \int_{\mathbb{R}^n} \frac{t^{1-\alpha}}{(t+|x-y|)^{n+1-\alpha}} |a_2(y)| dy \right]^2 \frac{dt}{t} w(x) dx d\theta. \]
For $x \in Q(s)$ and $y \in (Q^*(s))^c$, we have $(s + |x-y|) \approx |x-y| \approx (t + |x-y|)$. Thus, the expression in square brackets in (2.6) is dominated by a constant times
\[ \left( \frac{t}{s} \right)^{1-\alpha} \int_{\mathbb{R}^n} \frac{s^{1-\alpha}}{(s + |x-y|)^{n+1-\alpha}} |a_2(y)| dy \leq C \left( \frac{t}{s} \right)^{1-\alpha} ||a||_s, \]
where the last inequality follows by a slight variant of a classical argument of Fefferman Stein [FS] (see, e.g., [Stz, Lemma 2.2]). Lemma 2.3 may now be obtained by plugging this last expression into (2.6).

3. Proof of Theorem 1.4

The proof is based on ideas developed by Lewis and Murray in [LM, §3]. Our objective is to prove
\[ \int_{\mathbb{R}} \int_{0}^{\infty} |Q_{s} K_{\alpha} f(x)|^2 \frac{ds}{s^{1+2\alpha}} w(x) dx \leq C_{\alpha} \int_{\mathbb{R}} |f(x)|^2 w(x) dx, \]
where, without loss of generality, we assumed that $||a||_s = 1$ (recall that $A = I_{\alpha}a$, $a \in \text{BMO}$). Here, $w \in A_2$ if $\frac{1}{2} \leq \alpha < 1$, or $w \in A_1$ if $0 < \alpha < 1$. We smoothly truncate the kernel of $K_a$ as follows. Choose a radial $\varphi \in C_0^\infty$, $0 \leq \varphi \leq 1$, where $\varphi \equiv 1$ on $\{|x| < 100\}$ and $\varphi \equiv 0$ on $\{|x| > 101\}$. For fixed $s$, we write
\[ \frac{|A(x) - A(y)|^2}{|x-y|^{1+\alpha}} \equiv [A(x) - A(y)]^2 \left\{ |x-y|^{-1-\alpha} \varphi \left( \frac{|x-y|}{s} \right) + |x-y|^{-1-\alpha} \left( 1 - \varphi \left( \frac{|x-y|}{s} \right) \right) \right\} \]
\[ \equiv [A(x) - A(y)]^2 \left\{ j_s(x-y) + k_s(x-y) \right\}. \]
We consider first the term corresponding to $j_s$, which is essentially the same as $\theta_1$ in [LM, (3.10)] (the term $\theta_2$ in [LM] will not arise in the present argument and their term $\theta_3$ corresponds to $k_s$). The part of the left side of (3.1) corresponding to $j_s$ is crudely bounded by
\[ \int_{0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \psi_s(x-z)(A(z) - A(y))^2 j_s(z-y) |f(y)| dz dy \right|^2 w(x) dx \frac{ds}{s^{1+2\alpha}}. \]
Since convolution with $|\psi_s|$ is controlled by the Maximal Function, by Muckenhoupt's theorem the last expression is no larger than a constant times
\[ \int_{0}^{\infty} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (A(x) - A(y))^2 j_s(x-y) |f(y)| dy \right)^2 w(x) dx \frac{ds}{s^{1+2\alpha}}. \]
for all $w \in A_2$. In analogy with [LM, (3.15)–(3.17)], we apply Minkowski’s integral inequality to obtain the bound
\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{A(x) - A(y)^2}{|x - y|^{1+\alpha}} |f(y)| \left( \int_{|x-y|/101}^{\infty} \frac{ds}{s^{1+2\alpha}} \right)^{1/2} dy \right)^2 w(x) dx \leq C_\alpha \|C^2(|f|)|^2_{L^2,w},
\]
where $C_2$ is the second-order fractional commutator with kernel
\[
k^{(2)}(x, y) \equiv \frac{(A(x) - A(y))^2}{|x - y|^{1+2\alpha}}.
\]
But $C_2$ is bounded on unweighted $L^2$ by a result of Murray [M2]. Furthermore, since $A \in I_\alpha(BMO) \subseteq \text{Lip}_\alpha$, the kernel $k^{(2)}$ satisfies “standard” Calderon-Zygmund estimates, so $C_2$ is bounded on $L^2_w$, $w \in A_2$, by the usual arguments (see, e.g., Coifman and Fefferman [CF]).

We now turn to the part of (3.1) corresponding to $k_s$. We consider the kernel of the composition of $Q_s$ with the operator
\[
f \to \int [A(x) - A(y)]^2 k_s(x - y) f(y) dy.
\]
Since $Q_s 1 = 0$, this kernel equals
\[
\int_{\mathbb{R}} \psi_s(x - z)[A(z) - A(y)]^2 [k_s(z - y) - k_s(x - y)] \, dz + \int_{\mathbb{R}} \psi_s(x - z) [(A(z) - A(y))^2 - [A(x) - A(y)]^2] k_s(x - y) \, dz \equiv H_s(x, y) + L_s(x, y).
\]

The terms $H_s$ and $L_s$ correspond to $\sigma_1$ and $\sigma_2$ in [LM, (3.24) and (3.25)]. We treat $L_s$ first, and following [LM, (3.32)] we write
\[
(A(z) - A(y))^2 - (A(x) - A(y))^2 = (A(z) - A(x))^2 + 2(A(z) - A(x))(A(x) - A(y)).
\]
Since $\int \psi = 0$, the part of $L_s$ containing the second part of (3.5) equals twice
\[
Q_s A(x)[A(x) - A(y)] k_s(x - y).
\]
Recall that $A = I_\alpha a$, with $a \in BMO$. Plugging (3.6) into (3.1) in place of $Q_s K_s$, we obtain
\[
(3.7) \int_{\mathbb{R}} \int_{0}^{\infty} |\tilde{Q}_s a(x) C_{1,100s} f(x)|^2 \frac{ds}{s} w(x) dx,
\]
where $\tilde{Q}_s \equiv s^{-\alpha} Q_s I_s$ and $C_{1,100s}$ is the smoothly truncated first fractional commutator with kernel
\[
\frac{A(x) - A(y)}{|x - y|^{1+\alpha}} \left( 1 - \phi \left( \frac{|x - y|}{s} \right) \right).
\]
But $|\tilde{Q}_s a(x)|^2 w(x) \frac{ds}{s} dx$ is a weighted Carleson measure for all $w \in A_2$ (see [J, pp. 85–87]), so by a standard argument (3.7) is no larger than $\|N(C_{1,100s} f)\|^2_{L^2,w}$, where $N$ is the nontangential maximal operator $N g_s(x_0) \equiv$
Now \( C_1 \) is bounded on \( L^2 \) by [M1], and since the kernel \((A(x) - A(y))(x - y)^{-1-\alpha}\) satisfies "standard" Calderon-Zygmund estimates, the corresponding maximal singular integral

\[
C_1, f = \sup_{s > 0} |C_{1,100s} f|
\]

is bounded on \( L^2_w, w \in A_2 \). Thus, as in [LM, (3.39)], the nontangential maximal function \( N(C_{1,100s} f) \) is bounded on \( L^2_w \). In fact, the observation in [LM] holds for any Calderon-Zygmund operator \( T \) with "standard" kernel \( k(x, y) \), since for \(|x - x_0| < s\), we have

\[
|T_{100s} f(x)| \leq \int |k(x, y)\Phi(|x - y|/s) - k(x_0, y)\Phi(|x_0 - y|/s)| |f(y)| dy + T_s f(x_0),
\]

where \( \Phi = 1 - \varphi \). The first term on the right side of (3.8) is no larger than

\[
C \int \frac{s^\varepsilon}{(s + |x_0 - y|)^{n+\varepsilon}} |f(y)| dy \leq CM f(x_0),
\]

by the standard kernel conditions for \( k(x, y) \).

Next, we consider the part of \( L_s \) containing the first term in (3.5). We need to estimate

\[
|T_{100s} f(x)| \leq \int \int \psi_s(x - z)[A(z) - A(x)]^2 |f(y)| dy |dz| w(x) dx.
\]

Now, \(|k_s(x - y)| \leq c/(|h| + |x - y|)^{1+\alpha} \) for \(|h| < s\). Furthermore, \( A \in \text{Lip}_\alpha \), so by the change of variables \( z \to z + x \), and then \( z \to sz \), we have that (3.9) is bounded by

\[
\int \int \int |A(x + sz) - A(x)|^2 (P_{sz}[|f|](x)) |dz| w(x) dx,
\]

where \( P_t \) denotes convolution with the kernel \( t^{\alpha/2}/(t + |x|)^{1+\alpha} \). Now by Minkowski's integral inequality, the square root of (3.10) is no larger than

\[
\left( \int \int |A(x + sz) - A(x)|^2 (P_{sz}[|f|](x)) |dz| w(x) dx \right)^{1/2}.
\]

The desired estimate now follows by the change of variable \( s \to s/|z| \), and a standard argument using the weighted Carleson measure condition (2.4), and the fact that the nontangential maximal function \( N(P_t f) \) is bounded on \( L^2_w \), \( w \in A_2 \).

To conclude the proof of Theorem 1.4, it remains to consider \( H_s \) in (3.4). The part of (3.1) corresponding to this term is dominated by

\[
\int \int \int |\psi_s(x - z)|(A(z) - A(y))^2 \frac{s}{|z - y|^{2+\alpha}} \times \chi_{\{|z - y| > 99s\}}|f(y)| dy |dz| w(x) dx \frac{ds}{s^{1+\alpha}},
\]

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where we have applied the mean value theorem to $k_s$ and used the fact that $|x-z| < s$. For $w \in A_2$, again by Muckenhoupt's Theorem, this last expression is no larger than a constant times
\[
\int_0^\infty \int_\mathbb{R} \left( \int_\mathbb{R} \frac{(A(x) - A(y))^2}{|x-y|^{1+2\alpha}} \left( \frac{s}{|x-y|} \right)^{1-\alpha} \right. \\
\times \chi\{|x-y| > 99s\} |f(y)| \, dy \left. \right) ^2 w(x) \, dx \, \frac{ds}{s}.
\]
By Minkowski's integral inequality, we have the bound
\[
\int_\mathbb{R} \left( \int_\mathbb{R} \frac{(A(x) - A(y))^2}{|x-y|^{1+2\alpha}} |f(y)| \left( \int_0^{(x-y)/99} \left( \frac{s}{|x-y|} \right)^{2(1-\alpha)} \frac{ds}{s} \right)^{1/2} dy \right) ^2 w(x) \, dx \\
\leq C \|C_2(|f|)\|_{L^2,w}^2,
\]
and the theorem follows.

4. Extension to the single layer potential

Consider first the modified single layer potential
\[
S_\alpha f(x) = \int_\mathbb{R} W_\alpha(x, y) f(y) \, dy,
\]
where
\[
W_\alpha(x, y) = |x-y|^{\alpha-1} \exp \left\{ -\frac{(A(x) - A(y))^2}{|x-y|^{2\alpha}} \right\}.
\]

We have the following weighted version of [LM, Theorem 2] (see also [LM, Theorems 4 and 5]).

**Theorem 4.1.** Let $A = I_\alpha a$, and suppose $w \in A_2$ if $\frac{1}{2} \leq \alpha < 1$, or $w \in A_1$ if $0 < \alpha < 1$. Then
\[
\|S_\alpha - c_\alpha I_\alpha f\|_{L^p(L^q)} \leq C_\alpha (\|a\|_p^2 + \|a\|_q^2) \|f\|_{L^2,w}.
\]

**Proof of Theorem 4.1.** We will follow [LM, Theorem 2] and obtain the theorem by an easy modification of the proof of Theorem 1.4. The operator $S_\alpha - c_\alpha I_\alpha$ has kernel
\[
|x-y|^{\alpha-1} \exp \left\{ -\frac{(A(x) - A(y))^2}{|x-y|^{2\alpha}} \right\} - 1.
\]
The expression in curly brackets in (4.2) is no larger than a constant times $[A(x) - A(y)]^2 |x-y|^{-2\alpha}$, so if we multiply (4.2) by a smooth radial cut-off factor $\varphi(|x-y|/s)$, then we get a term that can be handled exactly like the term corresponding to $j_s$ in (3.2). It therefore remains to treat (4.2) times
\[
\Phi \left( \frac{|x-y|}{s} \right) \equiv \left( 1 - \varphi \left( \frac{|x-y|}{s} \right) \right).
\]
As in (3.4), we must consider the following analogues of (3.47) and (3.48) in [LM]:
\[
\int_\mathbb{R} \psi_s(x-z) \exp \left\{ -\frac{(A(z) - A(y))^2}{|z-y|^{2\alpha}} \right\} - 1 \right) \\
\times |z-y|^{\alpha-1} \Phi \left( \frac{|z-y|}{s} \right) - |x-y|^{\alpha-1} \Phi \left( \frac{|x-y|}{s} \right) \right) \, dz.
\]
weighted sobolev spaces, commutators, and layer potentials

\begin{equation}
\int_{\mathbb{R}} \psi_s(x-z) \left\{ \exp \left[ -\frac{(A(z) - A(y))^2}{|z - y|^{2\alpha}} \right] - \exp \left[ -\frac{(A(x) - A(y))^2}{|x - y|^{2\alpha}} \right] \right\} \times |x - y|^{n-1} \Phi \left( \frac{|x - y|}{s} \right) \, dz.
\end{equation}

These correspond to \( H_s \) and \( L_s \) in (3.4) respectively. The former can be handled exactly as before; in fact, we obtain the same upper bound (3.11).

Next, by Taylor's theorem the expression in curly brackets in (4.4) equals

\begin{equation}
- \left[ \frac{(A(z) - A(y))^2}{|z - y|^{2\alpha}} - \frac{(A(x) - A(y))^2}{|x - y|^{2\alpha}} \right] \exp \left[ -\frac{(A(x) - A(y))^2}{|x - y|^{2\alpha}} \right],
\end{equation}

\begin{equation}
+ \left[ \frac{(A(z) - A(y))^2}{|z - y|^{2\alpha}} - \frac{(A(x) - A(y))^2}{|x - y|^{2\alpha}} \right]^2 E(x, y, z),
\end{equation}

where \( 0 \leq E(x, y, z) \leq 1 \). By analogy to (3.4) and (3.5),

\begin{equation}
\frac{(A(z) - A(y))^2}{|z - y|^{2\alpha}} - \frac{(A(x) - A(y))^2}{|x - y|^{2\alpha}} = (A(z) - A(y))^2 \left[ \frac{1}{|z - y|^{2\alpha}} - \frac{1}{|x - y|^{2\alpha}} \right]
+ \frac{(A(z) - A(x))^2}{|x - y|^{2\alpha}} + \frac{2(A(z) - A(x))(A(x) - A(y))}{|x - y|^{2\alpha}}
\equiv B_1(x, y, z) + B_2(x, y, z) + B_3(x, y, z).
\end{equation}

Since \( |x - z| < s \ll |x - y| \) (so, in particular, \( |x - y| \approx |z - y| \)), we can handle the part of (4.5) corresponding to \( B_1 \) exactly like \( H_s \) in (3.4) (see (3.11)). Since (4.6) is no larger than \( CE(x, y, z) \sum_{i=1}^3 (B_i(x, y, z))^2 \), and since trivially

\[ |B_i| \leq C\|A\|_{\text{Lip}, \alpha} \leq C\|a\|^2, \]

the same reasoning applies to the parts of (4.6) corresponding to \( B_1 \). Similarly, those parts of (4.5) and (4.6) involving \( B_2 \) may be treated exactly like the first term in (3.5) (see (3.9), (3.10), and the related discussion). The latter argument also applies to the term \( (B_3(x, y, z))^2 E(x, y, z) \) arising in (4.6).

Thus, it remains only to consider the following part of (4.5): \( B_3(x, y, z) \times \exp[-(A(x) - A(y))^2/|x - y|^{2\alpha}] \) (we have ignored multiplication by \(-2\)). This is the only term where we do not reduce matters to the treatment of an appropriate positive operator, so the presence of a bounded, nonconstant multiplicative factor can no longer be ignored. If we plug this last expression into (4.4) in place of \{ \} and let the corresponding operator act on a function \( f \), then we get (since \( \int \psi = 0 \))

\[ Q_s A(x) T_{100s} f(x), \]

where \( T_{100s} \) has the (truncated) standard kernel

\[ \frac{(A(x) - A(y))}{|x - y|^{1+\alpha}} \exp \left[ -\frac{(A(x) - A(y))^2}{|x - y|^{2\alpha}} \right] \Phi \left( \frac{|x - y|}{s} \right). \]

As before (see (3.7), (3.8), and the related discussion), the theorem will follow by weighted Carleson measure theory once we show that the maximal singular integral

\[ T^* f \equiv \sup_{s > 0} |T_{100s} f| \]
is bounded on $L^2_w$, $w \in A_2$. But this is easy since the mean value theorem gives

$$\exp \left[ -\frac{(A(x) - A(y))^2}{|x - y|^{2\alpha}} \right] = 1 + \frac{(A(x) - A(y))^2}{|x - y|^{2\alpha}} \widetilde{E}(x, y),$$

with $|\widetilde{E}| \leq 1$. The term corresponding to 1 is just the first fractional commutator $C_1$, and the term corresponding to the second part of the right side of (4.8) is no larger than $C||A||_{\text{Lip},\alpha}C_2(||f||)$, and we are done.

In conclusion, we remark that as in [LM], a straightforward modification of the above arguments enables one to multiply the kernels that we have considered (e.g., (4.2) or (1.3)) by $\chi\{x - y > 0\}$. In particular, for $\alpha = \frac{1}{2}$, we can treat the boundary single layer potential for the heat equation for all $w \in A_2$ and thus for all $p$, $1 < p < \infty$.

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BIBLIOGRAPHY