FLAT CORE PROPERTIES ASSOCIATED TO THE $p$-LAPLACE OPERATOR

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Abstract. We study the formation of a flat hat pattern in the profile of the positive solution of an equation of the type: $\varepsilon \Delta_p u - u^{p-1}(1 - u)\theta = 0$ ($0 < \theta < p - 1$) in a bounded domain $\Omega$. When $\varepsilon$ tends to $0^+$, the growth of the zone where $u = u_\varepsilon$ takes the value 1 in $\Omega$ is studied.

Introduction and statement of the results

This paper deals with the study of the limit behaviour when $\varepsilon$ tends to $0^+$ of the shape of the positive solution $u = u_\varepsilon$ of the problem

\begin{equation}
-\varepsilon \Delta_p u + f(u) = 0 \quad \text{in } \Omega,
\end{equation}

\begin{equation}
u = 0 \quad \text{on } \partial \Omega,
\end{equation}

where $\Omega$ is a connected, bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with a $C^2$ boundary $\partial \Omega$; $\Delta_p$ is the $p$-Laplace operator defined by

\begin{equation}
\Delta_p u = \text{div}(\|\nabla u\|^{p-2}\nabla u)
\end{equation}

with $p > 1$; and $f$ is continuous with nonpositive values. Such a problem appears when studying the stationary states of a strongly nonlinear heat equation in an absorbing-reacting media (see [D] for physical examples and further references). The precise hypotheses on $f$ are the following:

(H1) $f$ is continuous on $[0, \infty)$ and $r \mapsto f(r)/r^{p-1}$ is increasing.

(H2) $\lim_{r \to 0} f(r)/r^{p-1} = -1$.

(H3) There exist $C > 0$ and $\theta \in (0, p - 1)$ such that $\lim_{r \to 1} f(r)/(1 - r)^\theta = -C$.

The specific phenomenon we shall study is the formation of a flat hat pattern inside $\Omega$, that is, a zone where $u$ takes the value 1 and the growth of this zone when $\varepsilon$ tends to 0.

The typical example of a function $f$ satisfying (H1)-(H3) is $f(u) = u^{p-1} - u^q$, thus problem (1) becomes

\begin{equation}
-\varepsilon \Delta_p u = u^{p-1} - u^q \quad \text{in } \Omega,
\end{equation}

\begin{equation}
u = 0 \quad \text{on } \partial \Omega.
\end{equation}

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If we set $\varepsilon = 1/\lambda$ and $u = v\varepsilon^{1/(q+1-p)}$, then (3) reads as

$$-\Delta_p v = \lambda v^{p-1} - v^q \quad \text{in } \Omega,$$

$$v = 0 \quad \text{on } \partial \Omega.$$  

The appearance of the flat zone for the solution of (4) for large $\lambda$ was first observed by Guedda and Veron. These authors in [GV] studied the structure of the set of solutions of the nonlinear eigenvalue problem

$$-(d|x|^{-2}v_x)_x = \lambda d|v|^{p-2}v - |v|^{q-1}v \quad \text{in } (0,1),$$

$$v(0) = v(1) = 0.$$  

It is proved in [GV] that for

$$q > p - 1 > 1$$

and $\lambda$ large enough, the unique positive solution of (5) satisfies $v(x) = \lambda^{1/(q+1-p)}$ for $x \in [x(\lambda), 1 - x(\lambda)]$ where $x(\lambda) > 0$ and $x(\lambda) \sim C\lambda^{-1/p}$ at infinity. Another consequence described in [GV] is that for $\lambda$ large enough, the set of solutions $v$ of (5) with $k - 1$ simple zeros on $(0,1)$ and $v_x(0) > 0$ is homeomorphic to the $(k-1)$-dimensional unit cube. P. L. Lions asked one of the authors whether such phenomenon still existed for the $N$-dimensional case.

If we define

$$\lambda_1 = \min \left\{ \int_{\Omega} |\nabla u|^p \, dx / \int_{\Omega} |u|^p \, dx : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\},$$

it is a classical fact that under condition (6), for any $\lambda > \lambda_1$ there exists $v$ positive in $\Omega$ satisfying (4). As for problem (1) we know from [DS] that if $\varepsilon < 1/\lambda_1$ and $f$ satisfies (H1), (H2), then there exists a unique $u - u_\varepsilon$ belonging to $C^1(\overline{\Omega})$ which is a positive in $\Omega$ solution of (1). Moreover if (H3) holds then $u$ takes its values in $[0, 1]$. If we define

$$\Omega_\lambda = \{ x \in \Omega : v(x) \equiv \lambda^{1/(q+1-p)} \}$$

then $\Omega_\lambda$ is a compact, possibly empty, subset of $\Omega$. We have the following answer to Lions's question

\textbf{Theorem 1.} Assume (6), $\lambda > \lambda_1$, $v$ is the positive solution of (4), and $\Omega_\lambda$ is defined by (8). Then there exists $\lambda^* = \lambda^*(\Omega, p, q)$ such that:

(i) if $\lambda < \lambda^*$ the set $\Omega_\lambda$ is empty;

(ii) if $\lambda \geq \lambda^*$ the set $\Omega_\lambda$ is not empty and

$$\text{dist}(\Omega_\lambda, \partial \Omega) \leq C\lambda^{-1/p},$$

where $C = C(\Omega, p, q) > 0$.

Theorem 1 is a consequence of

\textbf{Theorem 2.} Assume (H1)–(H3) with $p > 1$. Then for $\varepsilon > 0$ small enough the coincidence set $\Omega_\varepsilon$ of the solution $u$ of (1) defined by

$$\Omega_\varepsilon = \{ x \in \Omega : u(x) = 1 \}$$

is not empty and there exists a constant $C > 0$ such that

$$\text{dist}(\Omega_\varepsilon, \partial \Omega) \leq Ce^{1/p}.$$
PROOFS OF THE RESULTS

We first extend the function \( f \) on \(( -\infty, 0)\) such that the resulting function defined on \( R \) is a continuous odd function. This function is still denoted by \( f \).

**Lemma 1.** Let \( w_1 \) and \( w_2 \) be two functions belonging to \( C(\overline{\Omega}) \cap W^{1,p}(\Omega) \) and such that

\[
0 = w_1 \leq w_2 \quad \text{on } \partial \Omega
\]
and

\[
w_1 \leq w_2,
\]
\[
-\Delta_p w_1 + f(w_1) \leq 0,
\]
\[
-\Delta_p w_2 + f(w_2) \geq 0
\]
in \( \Omega \). Then there exists a function \( w \) in \( C_0(\Omega) \cap W^{1,p}(\Omega) \) satisfying

\[
w_1 \leq w \leq w_2,
\]
\[
-\Delta_p w + f(w) = 0
\]
in \( \Omega \).

This result is due to Deuel and Hess [DeH] and extends previous results of Amann, Sattinger, and others (see [A] for example).

**Lemma 2.** Let \( w \in C(\overline{\Omega}) \cap W^{1,p}_0(\Omega) \) be a positive solution of (17) in \( \Omega \). Then for \( C > 1 \) (resp. \( 0 < C < 1 \)) we have

\[
-\Delta_p(Cw) + f(Cw) \geq 0 \quad \text{(resp. } -\Delta_p(Cw) + f(Cw) \leq 0)\] in \( \Omega \).

**Proof.** For \( C > 1 \) we have

\[
\Delta_p(Cw) = C^{p-1}\Delta_p w = C^{p-1}f(w) = (Cw)^{p-1}f(w)/w^{p-1}.
\]
From (H1) we have \( f(w)/w^{p-1} \leq f(Cw)/(Cw)^{p-1} \), which yields (18). The same proof applies for \( 0 < C < 1 \).

**Lemma 3.** Assume (Hi) \((i = 1, 2, 3)\) and let \( u = u_\varepsilon \) be the positive solution of (9). Then \( u_\varepsilon \) converges to 1 as \( \varepsilon \) tends to 0, uniformly on any compact subset \( K \) of \( \Omega \).

**Proof.** By the maximum principle, \( u_\varepsilon \leq 1 \) in \( \overline{\Omega} \). The intent of this proof is to construct a subsolution \( v \) of (9) with the form

\[
v = 1 - e^{-\psi'/\varepsilon'}, \quad \varepsilon' = \varepsilon^{1/p},
\]

with some \( \psi > 0 \) in \( \Omega \), vanishing on \( \partial \Omega \); the function \( \psi \) will be made precise later. Then

\[
\nabla v = \frac{1}{\varepsilon'}e^{-\psi'/\varepsilon'} \nabla \psi
\]
and

\[
-\Delta_p v = (\varepsilon')^{-p}e^{-(p-1)\psi'/\varepsilon'} \{(p-1)|\nabla \psi|^p - \varepsilon'\Delta_p \psi\},
\]
which yields

\[
-\varepsilon\Delta_p v + f(v) = [(p-1)|\nabla \psi|^p - \varepsilon'\Delta_p \psi]e^{-(p-1)\psi'/\varepsilon'} + f(v).
\]
Set $E_1 = (p-1)|\nabla \psi|^p$, $E_2 = -\epsilon'\Delta_p \psi$, and $y = e^{-\psi/\epsilon'}$. We claim that for $\epsilon'$ small enough

$$E_1 + E_2 \leq -y^{1-p} f(1-y).$$

For $\delta > 0$ we define

$$\Omega^\delta_- = \{x \in \Omega : y \leq \delta\} = \{x \in \Omega : \psi \geq \epsilon' \ln(1/\delta)\},$$

$$\Omega^\delta_+ = \{x \in \Omega : y > \delta\} = \{x \in \Omega : \psi < \epsilon' \ln(1/\delta)\}.$$

From (H3) $\lim_{y \to 0^+} (-y^{-\theta} f(1-y)) = C$; henceforth there exists $\delta_0 \in (0, 1)$ such that $-y^{-\theta} f(1-y) > C/2$ for $0 < y < \delta_0$, which implies

$$-\frac{1}{y^{p-1}} f(1-y) > \frac{C}{2\delta^{p-1-\theta}} \quad \forall y \in (0, \delta_0)$$

as $p - 1 - \theta > 0$. We shall take $\psi = \phi_1^p$ where $\phi_1$ is the unique positive solution with upper bound 1 of

$$-\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} \quad \text{in } \Omega,$$

$$\phi_1 = 0 \quad \text{on } \partial \Omega.$$

There exists $M > 0$ such that for any $\epsilon' \in (0, 1]$ we have

$$|(p-1)|\nabla \psi|^p - \epsilon'\Delta_p \psi| \leq M,$$

and there exists $\delta_1 \in (0, \delta_0]$ such that for $\delta < \delta_1$

$$(p-1)|\nabla \psi|^p - \epsilon'\Delta_p \psi \leq M \leq c/2\delta^{p-1-\theta}$$

in $\Omega$, which implies that (22) holds in $\Omega^\delta_-$. For the estimate in $\Omega^\delta_+$ note that there exist two positive constants $l(\delta)$ and $r(\delta)$ such that

$$-f(1-y) \geq l(\delta)(1-y)^{p-1} \quad \forall y \in (\delta, 1)$$

and, consequently,

$$-f(1-y) \geq r(\delta)(\psi/\epsilon')^{p-1}$$

if $\delta < y \leq 1$ or $\psi/\epsilon' < \ln(1/\delta)$; we used here that $(1-e^{-p})/p$ is bounded below on $(0, \ln(1/\delta))$. In order to have

$$E_1 \leq -f(1-y)/y^{p-1}$$

in $\Omega^\delta_+$, it is sufficient to assure (with $y \leq 1$) that

$$(p-1)|\nabla \psi|^p \leq r(\delta)(\psi/\epsilon')^{p-1}$$

or, equivalently,

$$(\epsilon')^{p-1}|\nabla \psi|^p \leq \frac{r(\delta)}{p-1} \psi^{p-1}.$$

As $\psi = \phi_1^p$ we have

$$(\epsilon')^{p-1}|\nabla \psi|^p \psi^{1-p} = (\epsilon')^{p-1} p^p |\nabla \phi_1|^p.$$

For $\delta \in (0, \delta_1)$ fixed, we can choose $\epsilon'_0 > 0$ such that (31) holds for $0 < \epsilon' \leq \epsilon'_0$. 

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For the remaining term we have
\[ \Delta_p \psi = \Delta_p \phi_1^p = p^{p-1}(p - 1)^2 \phi_1^{p-1} |\nabla \phi_1|^p - \lambda_1 p^{p-1} \phi_1^{p-1} \cdot \]
As \( \partial \phi_1 / \partial \nu < 0 \) on \( \partial \Omega \) there exists a neighborhood \( D \) of \( \partial \Omega \) such that
\[ (p - 1)^2 |\nabla \phi_1|^p > \lambda_1 \phi_1^{p-1} \cdot \]
in \( D \). For \( \delta \) fixed in \( (0, \delta_1) \) there exists \( \varepsilon'_1 \in (0, \varepsilon'_0) \) such that for any \( \varepsilon' \in (0, \varepsilon'_1) \), \( \Omega^\varepsilon \subset D \). For such a limitation on \( \varepsilon' \) we have \( \Delta_p \psi > 0 \) in \( \Omega^\varepsilon \), which implies
\[ E_1 + E_2 \leq -y^{1-p} f(1-y) \cdot \]
in \( \Omega^\delta \). Henceforth, with this restriction on \( \varepsilon' \), \( v \) satisfies
\[ -\varepsilon' \Delta_p v + f(v) \leq 0 \cdot \]
in \( \Omega \) and \( v \) vanishes on \( \partial \Omega \). Now we compare \( u \) and \( v \). By Vazquez’s maximum principle [V], \( \partial u / \partial \nu < 0 \) on \( \partial \Omega \); therefore, there exists \( C > 1 \) such that \( Cu \geq v \) in \( \Omega \). Using Lemmas 2 and 1 we get that there exists a solution \( u^* \) of (9) such that \( v \leq u^* \leq Cu \). By uniqueness \( u^* = u \geq v \). For any compact subset \( K \subset \Omega \), there exists \( \eta(K) > 0 \) such that \( \psi \geq \eta(K) \) on \( K \). Letting \( \varepsilon \) tend to 0 implies the claimed result.

**Proof of Theorem 2.** Let \( \hat{u} = \hat{u}_\varepsilon = 1 - u \). From (H3) there exists \( \delta_0 > 0 \) such that \( -\hat{u}^{-\theta} f(1 - \hat{u}) > c/2 \) for \( 0 < \hat{u} < \delta_0 \), which yields
\[ -\Delta_p \hat{u} + \frac{c}{2} \hat{u}^\theta \leq 0 \cdot \]
if \( 0 < \hat{u} \leq \delta_0 \). For \( \eta > 0 \) let \( K_\eta \) be the subset of the \( x \)'s in \( \Omega \) such that \( \text{dist}(x, \partial \Omega) \geq \eta \). From Lemma 3, for any \( \delta \in (0, \delta_0) \) there exists \( \varepsilon(\delta) > 0 \) such that for \( 0 < \varepsilon < \varepsilon(\delta) \) we have
\[ \max \{ \hat{u}_\varepsilon(x) : x \in K_\eta \} < \delta \cdot \]
Let \( h = h_\delta \) be the solution of
\[ -\Delta_p h + \frac{c}{2} h^\theta = 0 \text{ in } B_\eta(0), \]
\[ h = \delta \text{ on } \partial B_\eta(0). \]
We know from Diaz-Herrero’s paper [DH] (see also [D, p. 41]) that there exists \( \delta > 0 \) such that \( h_\delta(0) = 0 \). Let \( x_0 \in K_{2\eta} \). By comparison, \( \hat{u}(x) \leq h_\delta(x - x_0) \) for \( |x - x_0| < \eta \). Thus \( \hat{u}(x_0) = 0 \). Therefore \( \hat{u}(x) \equiv 0 \) on \( K_{2\eta} \).

In order to obtain the final estimate we use a local scaling argument. As \( \partial \Omega \) is \( C^2 \) there exists \( \rho > 0 \) such that for any \( a \in \partial \Omega \) the open ball with center \( a - \rho \vec{v}_a \) and radius \( \rho \) is included into \( \Omega \) (\( \vec{v}_a \) is the normal unit vector to \( \partial \Omega \) at \( a \)). As we already proved, there exists \( \varepsilon_1 > 0 \) such that the positive solution \( z \) of
\[ -\varepsilon_1 \Delta_p z + f(z) = 0 \text{ in } B_\rho(0), \]
\[ z = 0 \text{ on } \partial B_\rho(0), \]
is such that \( z(x) \equiv 1 \ \forall x \in B_{\rho/2}(0) \). For \( k > 0 \) the function \( z_k \) defined by \( z_k(x) = z(kx) \) satisfies
\[ -\varepsilon_1 k^{-p} \Delta_p z_k + f(z_k) = 0 \text{ in } B_{\rho/k}(0), \]
\[ z_k = 0 \text{ on } \partial B_{\rho/k}(0). \]
and is such that $z_k(x) \equiv 1 \forall x \in B_{p/2k}(0)$. For $0 < \varepsilon < \varepsilon_1$ let $k$ be $(\varepsilon_1/\varepsilon)^{1/p}$, $k > 1$. For any $a \in \Omega$ such that $\text{dist}(a, \partial\Omega) \geq \rho/k$ we can compare $u(x)$ and $z_k(x-a)$ in $B_{p/k}(a)$. By the same way as in the proof of Lemma 3, we use Lemmas 2 and 1 with $\alpha > 0$ small enough. We get

$$\alpha z_k(x-a) \leq u(x) \quad \text{in } B_{p/k}(a),$$

which implies

$$z_k(x-a) \leq u(x) \quad \text{in } B_{p/k}(a).$$

We deduce that $u \equiv 1$ in $B_{p/2k}(a)$, which implies (11).

Remark 1. It is clear that the coincidence set $\Omega_\varepsilon$ may be empty if $\varepsilon$ is too large. To have an estimate of this minimal $\varepsilon$ we can proceed as follows: let $d > 0$ be the infimum of the distance of two hyperplanes that are parallel and such that $\Omega$ is contained into the strip limited by them. As the equation (9) is equivariant with respect to rotations and translations in $\mathbb{R}^N$, we can assume that

$$\Omega \subset \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : 0 < x_1 < d\}.$$

Let $\tilde{\zeta}$ be the unique positive solution of

$$\begin{align*}
-\varepsilon(|\tilde{\zeta}_{x_1}|^{p-2}\tilde{\zeta}_{x_1})_{x_1} + f(\tilde{\zeta}) &= 0 \quad \text{in } (0, d), \\
\tilde{\zeta}(0) &= \tilde{\zeta}(d) = 0.
\end{align*}$$

It is clear that $\tilde{\zeta}(x) = \zeta(x_1)$ satisfies

$$\begin{align*}
-\varepsilon\Delta_p \tilde{\zeta} + f(\tilde{\zeta}) &= 0 \quad \text{in } \Omega, \\
\tilde{\zeta} &\geq 0 \quad \text{on } \partial\Omega.
\end{align*}$$

As before there exists a solution $\tilde{u}$ such that for some $\alpha < 1$

$$\alpha u \leq \tilde{u} \leq \zeta$$

and by uniqueness $\tilde{u} = u \leq \zeta$. If $0 < \zeta < 1$ in $(0, d)$ we deduce that the coincidence set $\Omega_\varepsilon$ is empty. In the particular case of equation (1) the coincidence set is empty if

$$\lambda^{1/p} d(q-1)/(q+1-p)$$

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[GV, Remark 2.3].

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