INTEGER-VALUED POLYNOMIALS, PRÜFER DOMAINS, AND LOCALIZATION

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Abstract. Let $A$ be an integral domain with quotient field $K$ and let $\text{Int}(A)$ be the ring of integer-valued polynomials on $A$: $\{P \in K[X]|P(A) \subseteq A\}$. We study the rings $A$ such that $\text{Int}(A)$ is a Prüfer domain; we know that $A$ must be an almost Dedekind domain with finite residue fields. First we state necessary conditions, which allow us to prove a negative answer to a question of Gilmer. On the other hand, it is enough that $\text{Int}(A)$ behaves well under localization; i.e., for each maximal ideal $m$ of $A$, $\text{Int}(A)_m$ is the ring $\text{Int}(A_m)$ of integer-valued polynomials on $A_m$. Thus we characterize this latter condition: it is equivalent to an "immediate subextension property" of the domain $A$. Finally, by considering domains $A$ with the immediate subextension property that are obtained as the integral closure of a Dedekind domain in an algebraic extension of its quotient field, we construct several examples such that $\text{Int}(A)$ is Prüfer.

Introduction

Throughout this paper $A$ is assumed to be a domain with quotient field $K$, and $\text{Int}(A)$ denotes the ring of integer-valued polynomials on $A$:

$$\text{Int}(A) = \{P \in K[X]|P(A) \subseteq A\}.$$ 

The case where $A$ is a ring of integers of an algebraic number field $K$ was first considered by Pólya [13] and Ostrowski [12]. In this case we know that $\text{Int}(A)$ is a non-Noetherian Prüfer domain [1, 4]. More generally, if $A$ is a Noetherian domain, $\text{Int}(A)$ is a Prüfer domain if and only if $A$ is a Dedekind domain with finite residue fields [5, Corollary 6.5].

In the general case, we have shown that if $\text{Int}(A)$ is a Prüfer domain, then $A$ is an almost Dedekind domain with finite residue fields [5, Proposition 6.3]. Recall that $A$ is an almost Dedekind domain if $A_m$ is a rank-one discrete valuation domain for each maximal ideal $m$ of $A$ [7]. The problem of determining conditions under which $\text{Int}(A)$ is Prüfer has not been resolved.

In [9] Gilmer shows that various classical examples of non-Noetherian almost Dedekind domains do not have the finite residue fields property, and hence $\text{Int}(A)$ is not Prüfer (as indeed he even proves that $\text{Int}(A) = A[X]$). However, using a theorem of Krull [11, Theorem 3] concerning extensions of valuations,
he gives a construction of non-Noetherian almost Dedekind domains with finite residue fields, yielding both examples where \( \text{Int}(A) \) is Prüfer and where it is not. Then he states two open questions, the second being closely related to his construction.

**Q4.** If \( A \) is an almost Dedekind domain such that \( \{|A/m| \mid m \in \text{Max}(A)\} \) is bounded, is \( \text{Int}(A) \) a Prüfer domain?

**Q5.** Suppose \( A_0 \) is a semilocal principal ideal domain with quotient field \( K_0 \) and \( K \) is an infinite algebraic extension of \( K_0 \) that is expressed as the union of a strictly ascending sequence \( \{K_i\} \) of finite algebraic extensions of \( K_0 \). Let \( A_i \) be the integral closure of \( A_0 \) in \( K_i \) and \( A \) the union of the \( A_i \). If \( \text{Int}(A) \) is a Prüfer domain, must there exist \( N \in \mathbb{N} \) such that, for all \( i, j \) with \( N < i < j \), \( \text{Int}(A_i) \subset \text{Int}(A_j) \)?

Throughout this paper we will generally assume that \( A \) is an almost Dedekind domain with finite residue fields and with quotient field \( K \); we will try to find necessary or sufficient conditions for \( \text{Int}(A) \) to be Prüfer.

In the first section we determine necessary conditions on every subfield \( K_0 \) of \( K \) such that \( K/K_0 \) is a countably generated algebraic extension. This allows us to answer Q4 negatively, but raises a new question:

**Q6.** Are these conditions sufficient?

In a second section we determine a sufficient condition: if \( A \) is an almost Dedekind domain with finite residue fields and if, for each maximal ideal \( m \) of \( A \), \( \text{Int}(A)_m = \text{Int}(A_m) \) (we will say that \( \text{Int}(A) \) behaves well under localization), then \( \text{Int}(A) \) is a Prüfer domain. This raises another question:

**Q7.** Is it necessary for \( \text{Int}(A) \) to behave well under localization to be Prüfer?

Next we state a necessary condition for good behaviour of localization called the **immediate subextension property**.

In the fourth section we restrict ourselves to the case where there is a subfield \( K_0 \) of \( K \) such that \( K/K_0 \) is a countably generated algebraic extension and the intersection \( A_0 = A \cap K_0 \) is a Dedekind domain with finite residue fields and quotient field \( K_0 \). We show that in this case the immediate subextension property is equivalent to good behaviour of localization.

In the following section we assume, moreover, that \( A \) is the integral closure of \( A_0 \) in \( K \). We then show that Gilmer’s condition in question Q5 is necessary and sufficient for \( \text{Int}(A) \) to behave well under localization. We show also that if \( K \) is a normal extension of \( K_0 \), then \( \text{Int}(A) \) is Prüfer if and only if \( A \) is an almost Dedekind domain with finite residue fields.

In the sixth and last section we give examples using Gilmer’s construction; the first one provides a negative answer to Q4, another shows that, when \( A_0 \) is not semilocal, the answer to Q5 may be negative, and yet another shows that the answers to Q6 and Q7 are not both affirmative.

1. **Necessary conditions**

First, recall that

1.1 [5, Proposition 6.3]. *If \( \text{Int}(A) \) is a Prüfer domain, then \( A \) is an almost Dedekind domain with finite residue fields.*
If \( m \) is a maximal ideal of an almost Dedekind domain \( A \), then \( A_m \) is the ring of a rank-one discrete valuation \( v_m \). If \( K_0 \) is a subfield of \( K \) such that \( K/K_0 \) is a countably generated algebraic extension, then the restriction of \( v_m \) to \( K_0 \) is a rank-one discrete valuation and the valuation ring of this restriction \( v_m|K_0 \) is \( A_m \cap K_0 \). Let \( e_m(K/K_0) \) be the ramification index of \( v_m \) over \( v_m|K_0 \).

Similar to Gilmer's residue field condition in question Q4:

\[(\alpha) \quad \{|A/m| | m \in \text{Max}(A)\} \text{ is bounded,}\]

we propose the following ramification condition:

\[(\beta) \quad \text{For any subfield } K_0 \text{ of } K \text{ such that } K/K_0 \text{ is a countably generated algebraic extension, } \{e_m(K/K_0) | m \in \text{Max}(A)\} \text{ is bounded.}\]

Neither condition is necessary in such a global version: for the first one consider \( A = \mathbb{Z} \) and observe that \( \text{Int}(\mathbb{Z}) \) is Prüfer; for the second one see Example 6.6. However, the next theorem shows that both weaker local versions are necessary; but each of them separately is not sufficient, and this will allow us to answer Gilmer's question Q4 negatively.

1.2. **Theorem.** Suppose \( \text{Int}(A) \) is a Prüfer domain. Let \( K_0 \) be a subfield of \( K \) such that \( K/K_0 \) is a countably generated algebraic extension. Then for each maximal ideal \( m \) of \( A \):

\[(a) \quad \{|A/n| | n \in \text{Max}(A), v_n|K_0 = v_m|K_0\} \text{ is bounded, and}\]

\[(b) \quad \{e_n(K/K_0) | n \in \text{Max}(A), v_n|K_0 = v_m|K_0\} \text{ is bounded.}\]

Condition (a) is also given by Gilmer [9, Theorem 13]. First let us recall two results:

1.3 [3, Corollary, p. 303]. For each maximal ideal \( m \) of \( A \), \( \text{Int}(A)_m \) is contained in the ring \( \text{Int}(A_m) \) of integer-valued polynomials on \( A_m \).

1.4 ([2, Proposition 2] or [4, Lemma 1]). Let \( V \) be the ring of a rank-one discrete valuation \( v \) with finite residue field of cardinal \( q \), and let \( P \) be an integer-valued polynomial on \( V \) of degree \( d > 0 \). Then \( v(P) > -d/(q-1) \), where \( v(P) \) denotes the infimum of the values of the coefficients of \( P \).

In fact, Lemma 1 of [4] shows that the \( A \)-module \( \text{Int}(A) \) is generated by polynomials \( Q_d(X) \), where \( \deg(Q_d) = d \) and \( v(Q_d) = \sum_{s>0}[d/q^s] \). Assertion 1.4 follows from the inequality \( \sum_{s>0}[d/q^s] < \sum_{s>0}(d/q^s) = d/(q-1) \), for each \( d > 0 \).

1.5. **Lemma.** Let \( A \) be an almost Dedekind domain and let \( K_0 \) be a subfield of \( K \) such that \( K/K_0 \) is an algebraic extension. Suppose there exists a rank-one discrete valuation \( v_0 \) of \( K_0 \) such that one of the following conditions fails:

\[(a) \quad \{|A/m| | m \in \text{Max}(A), v_m|K_0 = v_0\} \text{ is bounded.}\]

\[(b) \quad \{e_m(K/K_0) | m \in \text{Max}(A), v_m|K_0 = v_0\} \text{ is bounded.}\]

Then \( \text{Int}(A) \cap K_0[X] \) is contained in \( V_0[X] \), where \( V_0 \) is the valuation ring of \( v_0 \).

**Proof.** Let \( Q \) be a nonconstant polynomial belonging to \( \text{Int}(A) \cap K_0[X] \), let \( d \) be its degree, and choose a maximal ideal \( m \) of \( A \) such that \( v_m|K_0 = v_0 \) and either \( |A/m| > d \) or \( e_m(K/K_0) > d \) (according to the failing condition). Since \( Q(A) \subset A \), then \( Q(A_m) \subset A_m \) (1.3); we will show that \( v_m(Q) \geq 0 \).

If \( |A/m| \) is infinite, then clearly \( Q \) is in \( A_m[X] \) [3, Proposition 5, Corollary 2]; if not we let \( q = |A/m| \).
If $|A/m| > d$, then $v_m(Q) > -(d/q - 1) \geq -1$ (1.4), hence $v_m(Q) \geq 0$.

If $e_m(K/K_0) > d$, then $v_m(Q) > -(d/q - 1) \geq -d$, but $Q$ belongs to $K_0[X]$ and $v_m(Q)$ is a multiple of $e_m(K/K_0)$, hence again $v_m(Q) \geq 0$. In any case, $Q$ belongs to $A_m[X] \cap K_0[X] = V_0[X]$.

1.6. Lemma. Let $A$ be an almost Dedekind domain and let $K_0$ be a subfield of $K$ such that $K/K_0$ is a countably generated algebraic extension. Suppose there exists a maximal ideal $m$ of $A$ such that one of the following conditions fails:

(a) $\{|A/n| \mid n \in \text{Max}(A), v_n|K_0 = v_m|K_0\}$ is bounded.
(b) $\{e_n(K/K_0) \mid n \in \text{Max}(A), v_n|K_0 = v_m|K_0\}$ is bounded.

Then there exists a maximal ideal $n$ of $A$ such that $\text{Int}(A)$ is contained in $A_n[X]$.

Proof. Let $\{K_j\}$ be an ascending sequence of finite extensions of $K_0$ such that $\bigcup_j K_j = K$. Let $n_0 = m$ and let $v_0$ be the restriction to $K_0$ of the valuation associated to $m$. For each $j > 0$ we define a maximal ideal $n_j$ of $A$ in the following way: (i) one of the conditions (a) or (b) fails with respect to the extension $K/K_j$ and the ideal $n_j$; (ii) $v_j|K_{j-1} = v_{j-1}$ where $v_j$ denotes the restriction to $K_j$ of the valuation associated to $n_j$. We can define such a sequence of maximal ideals $n_j$ since, for each $j > 0$, there are only finitely many valuations of $K_j$ extending $v_{j-1}$.

Let $v$ be the rank-one valuation of $K$ whose restriction to each $K_j$ is $v_j$. Let $V$ be the valuation ring of $v$ and $V_j$ the valuation ring of $v_j$. By construction $A \cap K_j$ is contained in $V_j$ and $A = \bigcup_j (A \cap K_j)$ is contained in $\bigcup_j V_j = V$. Hence there is a maximal ideal $n$ of $A$ such that $A_n = V$ since $A$ is an almost Dedekind domain. Now let $Q$ be any element of $\text{Int}(A)$ and let $j$ be an integer such that $Q$ belongs to $K_j[X]$. Lemma 1.5 implies that $Q$ belongs to $V_j[X]$ and hence $Q$ belongs to $A_n[X]$.

Proof of Theorem 1.2. If $\text{Int}(A)$ is Prüfer, then $A$ is an almost Dedekind domain (1.1). Suppose there exists a subfield $K_0$ of $K$ such that $K/K_0$ is a countably generated algebraic extension and there exists a maximal ideal $m$ of $A$ such that one of conditions (a) or (b) fails. Then $\text{Int}(A)$ is contained in $A_n[X]$ for some maximal ideal $n$ of $A$ (Lemma 1.6). Every overring of the Prüfer domain $\text{Int}(A)$ is Prüfer; but $A_n[X]$ is not Prüfer since $A_n$ is not a field. This is a contradiction.

Note that if conditions (a) and (b) of Theorem 1.2 are satisfied for a subfield $K_0$ of $K$, then they are also satisfied for any subfield $K_1$ of $K$ containing $K_0$. In the last paragraph we give examples of almost Dedekind domains $A$, which are integral extensions of the discrete valuation domain $\mathbb{Z}_p$ (with $p = 2$); hence $K_0 = Q$ is the smallest subfield of $K$. In Example 6.2, condition (a) is satisfied (and even condition $(\alpha)$ of Gilmer), but not (b); in Example 6.3 condition (b) is satisfied (and even condition $(\beta)$) but not (a). Therefore the domains $\text{Int}(A)$ are not Prüfer. Thus neither condition $(\alpha)$ nor $(\beta)$ separately is sufficient, and in particular,

1.7. Corollary. The answer to question $Q_4$ is negative.

1.8. Question $Q_6$. Let $A$ be an almost Dedekind domain with quotient field $K$ and $K_0$ a subfield of $K$ such that $K/K_0$ is an infinite countably generated
algebraic extension. If for each maximal ideal \( m \) of \( A \):

- \( \{ |A/n| \mid n \in \text{Max}(A), v_n|K_0 = v_m|K_0 \} \) is bounded, and
- \( \{ e_n(K/K_0) | n \in \text{Max}(A), v_n|K_0 = v_m|K_0 \} \) is bounded,

is \( \text{Int}(A) \) a Prüfer domain?

2. Localization

Now, in order to get sufficient conditions, we consider localization properties.

2.1. **Theorem.** Let \( A \) be an almost Dedekind domain with finite residue fields. If, for each maximal ideal \( m \) of \( A \), \( \text{Int}(A)_m = \text{Int}(A_m) \), then \( \text{Int}(A) \) is a Prüfer domain.

**Proof.** We have to show that, for each maximal ideal \( \mathfrak{P} \) of \( \text{Int}(A) \), \( \text{Int}(A)_\mathfrak{P} \) is a valuation domain. If \( \mathfrak{P} \cap A \) is a maximal ideal \( m \) of \( A \), then \( \text{Int}(A)_\mathfrak{P} = (\text{Int}(A)_m)_\mathfrak{P} \), and it is enough to show that \( \text{Int}(A)_m \) is a Prüfer domain. If \( \mathfrak{P} \cap A = (0) \), then, for any maximal ideal \( m \) of \( A \), \( \text{Int}(A)_\mathfrak{P} = (\text{Int}(A)_m)_\mathfrak{P} \), and it is also enough to show that \( \text{Int}(A)_m \) is a Prüfer domain. This results from 2.2 below, since, for each maximal ideal \( m \) of \( A \), \( A_m \) is a rank-one discrete valuation domain with finite residue field and \( \text{Int}(A)_m = \text{Int}(A_m) \) is Prüfer.

2.2  **[4, Propositions 1 and 2].** If \( V \) is a rank-one discrete valuation domain with finite residue field and with quotient field \( K \), then:

- (i) the prime ideals of \( \text{Int}(V) \) lying over the ideal \( (0) \) of \( V \) are the ideals \( \mathfrak{P}_S = \{ Q \in \text{Int}(V) | Q = S \cdot R, R \in K[X] \} \) where \( S \) is an irreducible polynomial of \( K[X] \);
- (ii) the prime ideals of \( \text{Int}(V) \) lying over the maximal ideal \( m \) of \( V \) are the maximal ideals \( m_\alpha(V) = \{ Q \in \text{Int}(V) | Q(\alpha) \in mV^* \} \), where \( V^* \) is the completion of \( V \) and \( \alpha \) is any element of \( V^* \);
- (iii) the localization of \( \text{Int}(V) \) with respect to \( \mathfrak{P}_S \) is the valuation ring \( K[X]_{(S)} \) and the localization of \( \text{Int}(V) \) with respect to \( m_\alpha(V) \) is the valuation ring \( V_\alpha = \{ R \in K(X) | R(\alpha) \in V^* \} \).

We will say that \( \text{Int}(A) \) behaves well under localization if the condition of Theorem 2.1 is fulfilled. Note that it is always the case when \( A \) is Noetherian [3, p. 303]; this yields the well-known fact:

2.3  **[5, Corollary 6.5].** If \( A \) is Noetherian, \( \text{Int}(A) \) is Prüfer if and only if \( A \) is a Dedekind domain with finite residue fields.

We know that \( \text{Int}(A) \) does not always behave well under localization when \( A \) is an almost Dedekind domain with finite residue fields [9, Theorem 13]. Conversely we ask:

2.4. **Question Q7.** For \( \text{Int}(A) \) to be Prüfer, is it necessary that it behaves well under localization?

2.5. **Remark.** If \( \text{Int}(A) \) is a Prüfer domain, then, for each maximal ideal \( m \) of \( A \) and each element \( \alpha \) of the completion \( A_m^* \) of \( A_m \), the localization of \( \text{Int}(A) \) with respect to \( m_\alpha = \{ Q \in \text{Int}(A) | Q(\alpha) \in mA_m^* \} \) is the valuation domain \( V_\alpha = \{ R \in K(X) | R(\alpha) \in A_m^* \} \). To see this, note that the domain \( \text{Int}(A) \) is contained in \( \text{Int}(A_m) \) (1.3) and the ideal \( m_\alpha \) is the intersection of \( \text{Int}(A) \) with the ideal \( (mA_m)\alpha \) of \( \text{Int}(A_m) \). If \( \text{Int}(A) \) is a Prüfer domain, then the localization
of $\text{Int}(A)$ with respect to $m_{\alpha}$ is a valuation domain. This latter domain is contained in the valuation domain $V_{\alpha}$ and its maximal ideal is contained in the maximal ideal of $V_{\alpha}$. Thus these valuation domains coincide. So when $\text{Int}(A)$ is Prüfer, every localization of $\text{Int}(A_{m})$ is a localization of $\text{Int}(A)$. Thus an equivalent form of Q2 asks whether every prime ideal of $\text{Int}(A)$ lying over a maximal ideal $m$ of $A$ is an ideal $m_{\alpha}$.

3. IMMEDIATE SUBEXTENSION PROPERTY

Now we state a necessary condition for $\text{Int}(A)$ to behave well under localization. Recall that the prime field of a field $K$ is the smallest subfield contained in $K$; it is isomorphic to $\mathbb{Q}$ or to $\mathbb{F}_{p}$.

3.1. Proposition. Let $m$ be a maximal ideal of $A$ such that $A_{m}$ is a rank-one discrete valuation domain with finite residue field. If $\text{Int}(A)_{m} = \text{Int}(A_{m})$, then there exists a subfield $K_{1}$ of $K$, finitely generated over the prime field of $K$, such that, if $A_{1} = A \cap K_{1}$ and $m_{1} = m \cap K_{1}$, then for each maximal ideal $n$ of $A$ lying over $m_{1}$,

$$nA_{n} = m_{1}A_{n} \text{ and } A/n \cong A_{1}/m_{1}.$$

Proof. We first construct the field $K_{1}$. Let $a_{0}, \ldots, a_{q-1}$ be a complete set of residues of $m$ in $A$ and let $t$ be a local parameter of $v$, which belongs to $A$. The polynomial $P = (X - a_{0}) \cdots (X - a_{q-1})/t$ belongs to $\text{Int}(A_{m})$ and also to $\text{Int}(A)_{m}$ by hypothesis. Let $s$ be an element of $A \setminus m$ such that $sP$ belongs to $\text{Int}(A)$ and define $K_{1}$ to be the subfield of $K$ generated over $\mathbb{Q}$ or $\mathbb{F}_{p}$ by $a_{0}, \ldots, a_{q-1}, t$, and $s$.

If we set $A_{1} = A \cap K_{1}$ and $m_{1} = m \cap K_{1}$, then $A_{1}/m_{1} \cong A/m$ and $tA_{1} \subset m_{1}$. Let $n$ be a maximal ideal of $A$ such that $n \cap A_{1} = m \cap A_{1} = m_{1}$. Let $a$ be any element of $A$. Since $sP(a)$ belongs to $A$, $stP(a) = s(a - a_{0}) \cdots (a - a_{q-1})$ belongs to $tA \subset n$. The element $s$ of $A_{1}$ does not belong to $m_{1}$, hence it does not belong to $n$, and there exists $i$ such that $a - a_{i}$ belongs to $n$. Therefore $A/n \cong A_{1}/m_{1}$. Let $b$ be any element of $n$. Since $sP(b + a_{0})$ is in $A$, $stP(b + a_{0}) = sb(b + a_{0} - a_{i}) \cdots (b + a_{0} - a_{q-1})$ is in $tA$; but $s(b + a_{0} - a_{i}) \cdots (b + a_{0} - a_{q-1})$ is in the multiplicative system $A \setminus n$, hence $b$ belongs to $tA_{n}$. Therefore $m_{1}A_{n}$ contains $n$ and $m_{1}A_{n} \cong nA_{n}$.

Recall that a valuation $v$ of $K$ is essential for the domain $A$ if the valuation ring of $v$ is the localization of $A$ with respect to a prime ideal $m$. Recall also that an extension $v$ of a valuation $v_{1}$ of a field $K_{1}$ is an immediate extension if $v$ and $v_{1}$ have the same value group and same residue field; in this case we will say that $v$ is immediate over $K_{1}$.

If $A_{n}$ is the ring of a valuation $v$, letting $v_{1}$ be the restriction of $v$ to $K_{1}$, the conditions $nA_{n} = m_{1}A_{n}$ and $A/n \cong A_{1}/m_{1}$ of Proposition 3.1 imply that $v$ is immediate over $K_{1}$. We then make the following definitions:

3.2. Definitions. Let $A$ be a Prüfer domain with quotient field $K$.

(i) A valuation $v$ of $K$, which is essential for $A$, is said to be totally $A$-immediate over a subextension $K_{1}$ if, letting $v_{1}$ be the restriction of $v$ to $K_{1}$, each extension $w$ of $v_{1}$ to $K$, which is essential for $A$, is immediate over $K_{1}$.

(ii) The domain $A$ is said to have the immediate subextension property over a subfield $K_{0}$ of $K$ if, for each valuation $v$, which is essential for $A$, there
exists a subextension $K_1$ of $K$ finitely generated over $K_0$, over which $v$ is totally $A$-immediate.

If the valuation $v$ of $K$ is totally $A$-immediate over a subfield $K_0$, then $v$ is totally $A$-immediate over every subfield $K_1$ containing $K_0$, and, if $A$ has the immediate subextension property over a subfield $K_0$ of $K$, then $A$ has the immediate subextension property over every subfield $K_1$ of $K$ containing $K_0$.

3.3. **Theorem.** Suppose $A$ is an almost Dedekind domain with finite residue fields. If, for each maximal ideal $m$ of $A$, $\text{Int}(A)_m = \text{Int}(A_m)$, then $A$ has the immediate subextension property over the prime field of $K$.

The theorem results from Proposition 3.1.

4. **Partial converse**

We now restrict ourselves to the case where there is a subfield $K_0$ of $K$ such that $K/K_0$ is a countably generated algebraic extension and the intersection $A_0 = A \cap K_0$ is a Dedekind domain with finite residue fields and quotient field $K_0$. We will show that the immediate subextension property is then equivalent to good behaviour of localization; we start with a lemma under somewhat more general conditions.

4.1. **Lemma.** Let $A$ be a Prüfer domain with quotient field $K$. Suppose that $K_0$ is a subfield of $K$ such that $K/K_0$ is a countably generated algebraic extension. The following assertions are equivalent:

(i) $A$ has the immediate subextension property over $K_0$.

(ii) For each valuation $v_0$ of $K_0$, which is the restriction of a valuation essential for $A$, there exists a finite extension $K'$ of $K_0$ contained in $K$ such that each extension of $v_0$ to $K$, which is essential for $A$, is immediate over $K'$.

**Proof.** It is clear that (ii) implies (i). Now suppose (ii) does not hold and write $K$ as the union of an ascending sequence of finite extensions $K_n$ of $K_0$. Then (ii) does not hold for at least one extension $v_1$ of $v_0$ to $K_1$, which is the restriction of an essential valuation for $A$; to see this, assume to the contrary that, for each extension $w$ of $v_0$ to $K_1$, which is the restriction of an essential valuation for $A$, there is an integer $j(w)$ such that each extension of $w$ to $K$, essential for $A$, is immediate over $K_j$. By induction it is then possible to construct a sequence $(v_n)$ of valuations, $v_n$ extending $v_{n-1}$ to $K_n$, such that there exists an extension $w_n$ of $v_n$ to $K$, essential for $A$, which is not an immediate extension of $v_n$. This sequence defines a valuation $v$ of $K$, extending $v_0$, and $v$ is essential for $A$. Indeed if $a$ is an element of $A$, then it belongs to a field $K_n$, and by construction $v(a) = v_n(a) = w_n(a) \geq 0$. Hence the ring of $v$ contains $A$. Now for each $n$, $v$ is not totally $A$-immediate over $K_n$, hence it is not totally $A$-immediate over any finite extension of $K_0$.

4.2. **Remark.** If $A$ is an almost Dedekind domain, it follows from Lemma 4.1 that the immediate subextension property over a subfield $K_0$ of $K$ such that $K/K_0$ is a countably generated algebraic extension is stronger than conditions (a) and (b) of Theorem 1.2.
We are now ready for the main theorem of this section.

4.3. Theorem. Let $A$ be an almost Dedekind domain with finite residue fields. If the quotient field $K$ of $A$ is a countably generated algebraic extension of a field $K_0$ and if the ring $A_0 = A \cap K_0$ is a Dedekind domain with finite residue fields and with quotient field $K_0$, then the following conditions are equivalent:

(i) For each maximal ideal $m$ of $A$, $\text{Int}(A)_m = \text{Int}(A_m)$.

(ii) $A$ has the immediate subextension property over $K_0$.

(iii) $A$ has the immediate subextension property over the prime field of $K$.

Proof. Theorem 3.3 shows that (i) implies (iii), and it is trivial that (iii) implies (ii). Conversely, if (ii) holds, let $m$ be a maximal ideal of $A$ and $P$ a polynomial in $\text{Int}(A_m)$. We have to show that $P$ belongs to $(\text{Int}(A))_m$, i.e., that $P = Q/s$, where $s$ belongs to $A \setminus m$ and $Q$ to $\text{Int}(A)$.

Let $K^0$ be a finite extension of $K_0$ such that $P$ belongs to $K^0[X]$. Let $A^0$ be the domain $A \cap K^0$; $A^0$ is a Dedekind domain since it contains the integral closure of $A_0$ in $K^0$, which is a Dedekind domain. Let $d$ be a nonzero element of $A^0$ such that the coefficients of $dP$ are in $A^0$. Then $d$ belongs to only a finite number of maximal ideals $p_k$ of $A^0$. If no $p_k$ is contracted from a maximal ideal of $A$, then $d$ is a unit of $A$ and the desired conclusion holds. For each $p_k$ lying under some maximal ideal of $A$, let $K(p_k)$ be a finite extension of $K^0$ such that each extension of the valuation associated to $p_k$ in $K^0$ to a valuation on $K$ that is essential for $A$ is immediate over $K(p_k)$ (Lemma 4.1). Define $K^*$ to be the finite extension of $K^0$ generated by these fields $K(p_k)$.

Let $A^*$ be the Dedekind domain $A \cap K^*$ and let $m^*$ be the maximal ideal $m \cap K^*$ of $A^*$. The localization of $A^*$ with respect to $m^*$ is the intersection of $A_m$ with $K^*$. By hypothesis $P(A_m)$ is contained in $A_m$ so that $P(A^*_{m^*})$ is contained in $A_m \cap K^* = (A^*)_m$ and $P$ belongs to $\text{Int}(A^*_{m^*})$. But $\text{Int}(A^*_{m^*}) = \text{Int}(A^*)_{m^*}$, because $A^*$ is Noetherian [3, Corollary 5, p. 303]; hence $P = Q/s$, where $s$ belongs to $A^* \setminus m^*$ and the polynomial $Q$ is such that $Q(A^*) \subset A^*$.

It remains to prove that $Q(A) \subset A$, or equivalently, that for each maximal ideal $n$ of $A$, $Q(A) \subset A_n$:

If $d \notin n$, then $dQ = dsP$ has its coefficients in $A$, and clearly $Q(A) \subset A_n$.

If $d \in n$, then let $n^* = n \cap A^*$. By construction, $nA_n = n^*A_n$ and $A/n \cong A^*/n^*$. From $Q(A^*) \subset A^*$ it follows that $Q(A^*) \subset A^*$. [3], and this implies that $Q(A_n) \subset A_n$ (Proposition 4.4 below).

4.4. Proposition [6, Proposition 5.5]. Let $R$ be a Noetherian local domain with maximal ideal $m$ and quotient field $K$. Let $R_0$ be a local subring of $R$ with maximal ideal $m_0$. If $m_0R = m$ and $R/m \cong R_0/m_0$, then

$$\text{Int}(R) = \{ P \in K[X] | P(R_0) \subset R \}.$$
Thus, with the hypothesis of Theorem 4.3, we have proved the following implications:

<table>
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4.5. **Corollary.** Let $A$ be an integrally closed domain such that the quotient field $K$ of $A$ is a countably generated algebraic extension of a field $K_0$ and the ring $A_0 = A \cap K_0$ is a Dedekind domain with finite residue fields and with quotient field $K_0$. If $A$ has the immediate subextension property over $K_0$, then:

(i) $A$ is an almost Dedekind domain with finite residue fields.
(ii) For each maximal ideal $m$ of $A$, $\text{Int}(A)_m = \text{Int}(A_m)$.
(iii) $\text{Int}(A)$ is a Prüfer domain.

**Proof.** In view of Theorem 4.3 and the diagram above, it suffices to prove (i). Note that $A$ is a Prüfer domain since $A$ is an overring of the integral closure of $A_0$ in $K$, which is a Prüfer domain. The immediate subextension property then implies that $A$ is an almost Dedekind domain with finite residue fields.

5. **Integral closure of a Dedekind domain**

Let $A_0$ be a Dedekind domain with finite residue fields, $K_0$ the quotient field of $A_0$, $K$ a countably generated algebraic extension of $K_0$, and $A$ the integral closure of $A_0$ in $K$. Under these hypotheses $A$ is a Prüfer domain, and if $A$ has the immediate subextension property, it is an almost Dedekind domain with finite residue fields.

5.1. **Proposition.** Let $A_0$ be a Dedekind domain with quotient field $K_0$, $K$ a countably generated algebraic extension of $K_0$, and $A$ the integral closure of $A_0$ in $K$. Suppose that $A$ is an almost Dedekind domain with finite residue fields. The following conditions are equivalent:

(i) For each maximal ideal $m$ of $A$, $\text{Int}(A)_m = \text{Int}(A_m)$.
(ii) $A$ has the immediate subextension property over $K_0$.
(iii) For each valuation $v_0$ of $K_0$, which is essential for $A_0$, there exists a finite extension $K_1$ of $K_0$ such that each extension $v$ of $v_0$ to $K$, which is essential for $A$, is immediate over $K_1$.

**Proof.** The equivalence (i) $\iff$ (ii) results from Theorem 4.3. Lemma 4.1 shows the equivalence (ii) $\iff$ (iii) since each valuation $v$ of $K$, which is essential for $A$, is the extension of—and has for restriction—a valuation $v_0$ of $K_0$, which is essential for $A_0$.

With regard to localization, the following corollary provides an affirmative answer to a question that is analogous to Gilmer’s question $Q_5$ quoted in the introduction.

5.2. **Corollary.** Let $A_0$ be a semilocal principal ideal domain with quotient field $K_0$. Let $K$ be the union of an ascending sequence of finite algebraic extensions
\( K_n \) of \( K_0 \). Let \( A \) be the integral closure of \( A_0 \) in \( K \) and, for each \( n \), let \( A_n = A \cap K_n \). If \( A \) is an almost Dedekind domain with finite residue fields, the following conditions are equivalent:

(i) For each maximal ideal \( m \) of \( A \), \( \text{Int}(A)_m = \text{Int}(A_m) \).

(ii) There exists \( n \) such that every valuation of \( K \), which is essential for \( A \), is immediate over \( K_n \).

(iii) There exists \( n \) such that for all \( i, j \) with \( n \leq i \leq j \), \( \text{Int}(A_i) \subset \text{Int}(A_j) \subset \text{Int}(A) \).

**Proof.** (i) \( \rightarrow \) (ii) results from Proposition 5.1 since \( A_0 \) is semilocal.

(ii) \( \rightarrow \) (iii) Let \( n \) be an integer as in condition (ii) and let \( i \) and \( j \) be such that \( n \leq i \leq j \). Let \( q \) be a maximal ideal of \( A_j \) and let \( p = q \cap A_i \). The valuation associated with the ring \( (A_j)_q \) is an immediate extension of the valuation associated with the ring \( (A_i)_p \); then

\[
\text{Int}(A_i) \subset \text{Int}(A_p) \subset \text{Int}(A_q) = \text{Int}(A) \]

for each maximal ideal \( q \) of \( A_j \) (Proposition 4.4). Therefore \( \text{Int}(A_i) \subset \text{Int}(A_j) \).

Moreover, \( \text{Int}(A_i) \subset \text{Int}(A) \); let \( Q \) be an element of \( \text{Int}(A_i) \). For each element \( a \) of \( A \), there exists \( j > i \) such that \( a \) belongs to \( A_j \); thus \( Q(a) \) belongs to \( A_j \) since \( \text{Int}(A_i) \subset \text{Int}(A_j) \), and \( Q \) is in \( \text{Int}(A) \).

(iii) \( \rightarrow \) (i) Let \( m \) be a maximal ideal of \( A \) and let \( Q \) be any element of \( \text{Int}(A_m) \). Let \( K_i \) be an extension of \( K_0 \) such that \( Q \) belongs to \( K_i[X] \) and let \( p \) be the ideal \( m \cap A_i \). By hypothesis \( Q(A_m) \) is contained in \( A_m \), so that \( Q((A_i)_p) \subset A_m \cap K_i = (A_i)_p \) and \( Q \) belongs to \( \text{Int}((A_i)_p) = \text{Int}(A_i)_p \). Hence \( Q = P/s \), where \( s \) is an element of \( A_i \setminus p \) and \( P \) belongs to \( \text{Int}(A_i) \). Since \( \text{Int}(A_i) \) is contained in \( \text{Int}(A) \), \( Q \) belongs to \( \text{Int}(A)_m \).

If \( K \) is a normal algebraic extension of \( K_0 \), the four properties in the diagram at the end of §4 are satisfied as soon as \( A \) is an almost Dedekind domain with finite residue fields and we have a characterization of when \( \text{Int}(A) \) is a Prüfer domain:

5.3. **Theorem.** Let \( A_0 \) be a Dedekind domain with finite residue fields and with quotient field \( K_0 \). Let \( K \) be a countably generated normal algebraic extension of \( K_0 \) and let \( A \) be the integral closure of \( A_0 \) in \( K \). The following conditions are equivalent:

(i) \( A \) is an almost Dedekind domain with finite residue fields.

(ii) \( A \) has the immediate subextension property over \( K_0 \).

(iii) \( \text{Int}(A) \) is a Prüfer domain.

(iv) For each maximal ideal \( m \) of \( A \):

(a) \( \{[A/n] \mid n \in \text{Max}(A), v_n|K_0 = v_m|K_0 \} \) is bounded, and

(b) \( \{e_n(K/K_0) \mid n \in \text{Max}(A), v_n|K_0 = v_m|K_0 \} \) is bounded.

If \( A_0 \) is semilocal, the equivalence of (i) and (iii) is also proved by Gilmer [9, Theorem 12].

**Proof.** Implications (ii) \( \rightarrow \) (iii) \( \rightarrow \) (iv) result from Theorems 2.1 and 1.2; (iv) \( \rightarrow \) (i) is immediate. Let us prove (i) \( \rightarrow \) (ii). Let \( v \) be a valuation of \( K \), which is essential for \( A \). Since \( v \) is discrete and has a finite residue field, there exists a finite extension \( K_1 \) of \( K_0 \) such that \( v \) is an immediate extension of
its restriction $v_1$ to $K_1$ ($K_1$ is generated by a complete set of residues and by a local parameter of $v$). As the extension $K/K_1$ is normal, every extension of $v_1$ to $K$ is also an immediate extension and $v$ is totally $A$-immediate over $K_1$.

6. Counterexamples

In order to answer some questions (and in particular Gilmer's question Q4) and shed some light on the others, we will give several examples. We construct rings of algebraic numbers, which are integral closures of $\mathbb{Z}$ or of localizations of $\mathbb{Z}$ in algebraic extensions of $\mathbb{Q}$. As Gilmer does, we use Hasse's existence theorem about prime ideal decomposition in algebraic number fields.

6.1 (Hasse [10]). Let $K_0$ be an algebraic number field and let $m_1, \ldots, m_s$ be prime ideals of the ring $A_0$ of algebraic integers of $K_0$. Suppose given, for each $i = 1, \ldots, s$, $2r(i)$ positive integers $e_{ij}$ and $f_{ij}$ ($j = 1, \ldots, r(i)$) in such a way that $\sum_{1 \leq j \leq r(i)} e_{ij} f_{ij} = n$ (for every $i = 1, \ldots, s$). Then there exists an algebraic extension $K$ of $K_0$ having degree $n$ such that each prime ideal $m_i$ decomposes in the field $K$ as a product $m_i = \prod_{1 \leq j \leq r(i)} M_{ij}$, where the $M_{ij}$ are prime ideals of the ring $A$ of algebraic integers of $K$ and $[A/M_{ij} : A_0/m_i] = f_{ij}$.

In every example that we are going to construct, $K$ is the union of a strictly ascending sequence of finite extensions $K_n$ of $K_0 = \mathbb{Q}$. We define every extension $K_n/K_{n-1}$ with Hasse's existence theorem, where $e_{ij} = 1$ or $f_{ij} = 1$.

6.2. Example. Let $v_0$ be the 2-adic valuation of $K_0 = \mathbb{Q}$ and $A_0 = \mathbb{Z}(2)$ the valuation ring of $v_0$. We define $K_n$ by induction on $n$: $K_n$ is an extension of $K_{n-1}$ such that (i) $[K_n : K_{n-1}] = n + 2$; (ii) every valuation of $K_{n-1}$, which is ramified over the valuation $v_0$ of $\mathbb{Q}$ has only immediate extensions to $K_n$ ($e = 1$, $f = 1$); (iii) the valuation $v_{n-1}$ of $K_{n-1}$, which is not ramified over $v_0$ has two extensions to $K_n$; one of these is totally ramified ($e = [K_n : K_{n-1}] - 1$, $f = 1$), and the other, denoted by $v_n$, is immediate ($e = 1$, $f = 1$).

Let $K$ be the union of the extensions $K_n$ and let $A$ be the integral closure of $A_0 = \mathbb{Z}(2)$ in $K$; $A$ is also the intersection of the valuation rings of the extensions of $v_0$ to $K$.

A tree may represent the extensions of valuations as follows: a single line represents an immediate extension, a multiple line represents a ramified extension (as depicted in the figure).
A valuation of \( K \) is a branch of the tree; each extension of \( v_0 \) has relative degree one and a finite ramification index. So \( A \) is an almost Dedekind domain and each residue field of \( A \) is isomorphic to \( \mathbb{F}_2 \), while \( \{v_m(2)\mid m \in \text{Max}(A)\} = \mathbb{N}^* \). The condition (b) of Theorem 1.2 does not hold and we answer Gilmer’s question \( Q_4 \) negatively:

\[
\text{Int}(A) \text{ is not a } \text{Prüfer domain and } A \text{ is an almost Dedekind domain such that } \{[A/m] \mid m \in \text{Max}(A)\} = \{2\} \text{ is bounded.}
\]

6.3. Example [9, Gilmer’s Example 14]. We reverse the roles of \( e \) and \( f \) in Example 6.2: every valuation of \( K_{n-1} \) whose residue field is not isomorphic to \( \mathbb{F}_2 \) has only immediate extensions to \( K_n \) (\( e = 1, f = 1 \)) and the valuation \( v_{n-1} \) of \( K_{n-1} \) whose residue field is isomorphic to \( \mathbb{F}_2 \) has two extensions to \( K_n \); one of these is such that \( e = 1 \) and \( f = [K_n : K_{n-1}] - 1 \), and the other, denoted by \( v_n \), is immediate. Then \( A \) is an almost Dedekind domain with finite residue fields. For each maximal ideal \( m \) of \( A \), \( v_m(2) = 1 \), while \( \{[A/m] \mid m \in \text{Max}(A)\} = \{2^n\mid n \in \mathbb{N}^*\} \). Condition (a) of Theorem 1.2 is not fulfilled and \( \text{Int}(A) \) is not a Prüfer domain, but condition (b) is satisfied. Thus the condition that \( \{v_m(m \cap A_0)\mid m \in \text{Max}(A)\} \) is bounded is not sufficient for \( \text{Int}(A) \) to be Prüfer.

Now let us show that the answer to question \( Q_5 \) would be negative if we did not hypothesize \( A_0 \) to be a semilocal domain.

6.4. Example. Let \( K_0 = \mathbb{Q} \) and \( A_0 = \mathbb{Z} \). Letting \( p_n \) be the \( n \)th prime number, we define \( K_n \) by induction on \( n \); \( K_n \) is an extension of \( K_{n-1} \) such that:

(i) \( [K_n : K_{n-1}] = 2 \); (ii) every valuation of \( K_{n-1} \), which is an extension of the 2-adic valuation or of the 3-adic valuation or..., the \( p_{n-1} \)-adic valuation of \( \mathbb{Q} \) is completely decomposed (has only immediate extensions to \( K_n \)); (iii) every valuation of \( K_{n-1} \) which is an extension of the \( p_n \)-adic valuation of \( \mathbb{Q} \), is totally ramified (has only one extension to \( K_n \) with \( e = [K_n : K_{n-1}] \) and \( f = 1 \)).

[Note that for each step we only consider a finite number of valuations.]

Let \( K \) be the union of the \( K_n \) and \( A \) the integral closure of \( \mathbb{Z} \) in \( K \). The ring \( A \) has the immediate subextension property: for every \( n \), each extension of the \( p_n \)-adic valuation is immediate over \( K_n \). Hence, for each maximal ideal \( m \) of \( A \), \( \text{Int}(A)_m = \text{Int}(A_0)_m \) and \( \text{Int}(A) \) is Prüfer (Corollary 4.5). However, for \( i < j \), each extension to \( K_j \) of the \( p_j \)-adic valuation is ramified over \( K_i \), hence \( \text{Int}(A_i) \) is not included in \( \text{Int}(A_j) \) ([6, Proposition 5.3] or [9, Proposition 11]).

Let us show now that conditions (a) and (b) of Theorem 1.2 do not imply that \( \text{Int}(A) \) behaves well under localization.

6.5. Example. A slightly different version of Example 6.2: \( [K_n : K_{n-1}] = 3 \). Then \( A \) is an almost Dedekind domain with finite residue fields since every extension of the 2-adic valuation \( v_0 \) has relative degree one and ramification index two, except the valuation \( v \), whose restriction to each \( K_n \) is \( v_n \) and which is an immediate extension of \( v_0 \). But, for each \( n \), \( v|K_n = v_n \) has ramified extensions to \( K \), and the immediate extension \( v \) of \( v_0 \) contradicts the immediate subextension property.
Hence Int(A) does not behave well under localization although conditions (a) and (b) of Theorem 1.2 are fulfilled. We do not know if Int(A) is Prüfer, but if Int(A) is Prüfer the answer to Q7 is negative and if Int(A) is not Prüfer the answer to Q6 is negative.

We said that Int(Z) is Prüfer although \{|Z/m||m \in \text{Max}(Z)\} is not bounded; let us construct now an example A such that Int(A) is Prüfer and \{v_m(m \cap Z)|m \in \text{Max}(A)\} is not bounded.

6.6. Example. A slightly different version of Example 6.4: \([K_n : K_{n-1}] = n+2\). The ring A has the immediate subextension property, hence Int(A) is Prüfer (Corollary 4.5) and conditions (a) and (b) of Theorem 1.2 are fulfilled, but condition (β) on ramification index is not.

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References


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