COMPLEMENTED SUBSPACES AND AMENABILITY: A COUNTEREXAMPLE

YUJI TAKAHASHI

(Communicated by J. Marshall Ash)

Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday

Abstract. We give an example of a left amenable discrete semigroup $S$ such that $l^\infty(S)$ has weak*-closed selfadjoint left translation invariant subalgebras that are weak*-complemented but not invariantly complemented in $l^\infty(S)$. This resolves negatively a problem raised by Lau.

Let $S$ be a (discrete) semigroup and let $l^\infty(S)$ be the Banach algebra of bounded complex-valued functions on $S$ with pointwise operations and supremum norm. Following [2] we say that a weak*-closed left translation invariant subspace $X$ of $l^\infty(S)$ is invariants complemented in $l^\infty(S)$ if $X$ admits a left translation invariant closed complement, or equivalently, $X$ is the range of a continuous projection on $l^\infty(S)$ commuting with left translations. In [3, Problem 3] Lau asked if left amenability of $S$ implies each of the following properties:

(C) Each weak*-closed left translation invariant complemented subspace of $l^\infty(S)$ is invariantly complemented in $l^\infty(S)$.

(C*) Each weak*-closed selfadjoint left translation invariant subalgebra of $l^\infty(S)$ is invariantly complemented in $l^\infty(S)$.

Notice that Lau’s argument in the proof of Theorem 3.3 in [2] shows that each of (C) and (C*) implies left amenability of $S$. We also note that each of (C) and (C*) is equivalent to left amenability of $S$ when $S$ is a group (see [2, Theorem 3.3; 4, Theorem 1]). The purpose of the present note is to give an example that shows the answer to Lau’s problem is negative. Our example also solves a question in [2, p. 232] negatively.

Recall that a weak*-closed subspace $X$ of $l^\infty(S)$ is called weak*-complemented in $l^\infty(S)$ if there exists a weak*-weak* continuous projection from $l^\infty(S)$ onto $X$. Our example is a direct consequence of the following result.

Received by the editors November 27, 1991.

1991 Mathematics Subject Classification. Primary 43A07, 43A15.

Key words and phrases. Left amenable semigroup, complemented subspace, invariantly complemented subspace, weak*-complemented subspace.
Theorem. Let $S$ be a semigroup and let $E$ be a subset of $S$ with the following properties:

(a) $E^c$ (the complement of $E$ in $S$) is a left ideal of $S$.
(b) There exists $t \in S$ such that $tS \cap E^c \cap (tE^c)^c$ is nonempty.

Let $X_E$ be the set of all functions of $l^\infty(S)$ that vanish on $E^c$. Then $X_E$ is a weak*-closed selfadjoint left translation invariant subalgebra of $l^\infty(S)$ that is weak*-complemented but not invariantly complemented in $l^\infty(S)$.

Proof. Obviously $X_E$ is a weak* -closed selfadjoint subalgebra of $l^\infty(S)$. It also follows immediately from the assumption (a) on $E$ and the definition of $X_E$ that $X_E$ is left translation invariant. For each $f \in l^\infty(S)$, define $Pf \in l^\infty(S)$ as follows: $Pf(s) = f(s)$ if $s \in E$ and $Pf(s) = 0$ if $s \in E^c$. Then $P$ is a continuous projection from $l^\infty(S)$ onto $X_E$. Furthermore, $P$ is weak*-weak* continuous because $P$ is the conjugate operator of the continuous linear operator $Q$ on $l^1(S)$ (the Banach space of all absolutely summable complex-valued functions on $S$) obtained by the same method as that $P$ was defined. Thus $X_E$ is weak*-complemented in $l^\infty(S)$.

Let us now give a typical example of a left amenable semigroup satisfying the assumption of our theorem.

Example. Let $S$ be the direct product of the additive semigroup $N$ of positive integers and a left amenable semigroup $T$. Since $N$ is abelian, $N$ is amenable and hence $S$ is also amenable [1, Theorems (17.5) and (17.18)]. For $n \in N$, let

$E(n) = \{(k, t) \in S : k \in N \text{ with } k \leq n \text{ and } t \in T\}$. 
Then $E(n)$ satisfies the assumption of our theorem for each $n$. Indeed, it is clear that $E(n)^c$ is a left ideal of $S$. Now choose and fix arbitrary elements $t$ and $u$ in $T$. Put $x = (1, t)$ and $y = (n, u)$. Then
\[ xy = (1 + n, tu) \in E(n)^c \cap E(n + 1). \]
Since $E(n + 1)$ is contained in $(xE(n)^c)^c$, it follows that $xy \in (xE(n)^c)^c$. Thus we have
\[ xy \in xS \cap E(n)^c \cap (xE(n)^c)^c, \]
and hence $E(n)$ satisfies assumption (b) of the Theorem. Consequently, for each $n$,
\[ X_{E(n)} = \{ f \in L^\infty(S) : f = 0 \text{ off } E(n) \} \]
is a weak*-closed selfadjoint left translation invariant subalgebra of $L^\infty(S)$ that is weak*-complemented but not invariantly complemented in $L^\infty(S)$. This resolves Lau’s problem [3, Problem 3] negatively. We also see that the answer to Lau’s problem is negative even when $S$ is abelian.

**Remark.** In [2, p. 232] Lau raised the question: If $S$ is a left amenable semigroup and if $X$ is a weak*-complemented left translation invariant weak*-closed subspace of $L^\infty(S)$, then is $X$ invariantly complemented? Notice that the answer is affirmative when $S$ is a group (see [2, Corollary 4.2]). Our example gives a negative answer to this question.

**References**