BOUNDARY BEHAVIOR OF GENERALIZED POISSON INTEGRALS
FOR THE HALF-SPACE AND THE DIRICHLET PROBLEM
FOR THE SCHRÖDINGER OPERATOR

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Abstract. The boundary properties are investigated for the generalized Poisson integral

\[ u(X) = \int_{\mathbb{R}^n} k(X, y)f(y)\,dy, \]

where \( X \) is a point of the upper half-space \( \mathbb{R}_+^{n+1} \), \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), and the kernel \( k \) has some special properties. Our results imply the known boundary properties of the harmonic Poisson integrals on the half-space. As an application we derive a solution of the Dirichlet problem for the operator \(-\Delta + c(\cdot)\), \( X \in \mathbb{R}_+^{n+1} \), with boundary data \( f \in L^p(\mathbb{R}^n) \).

1. Introduction and statement of results

In their well-known paper [1] Fefferman and Stein extended the classical theory of Hardy spaces \( H^p \) on harmonic functions in the half-space

\[ \mathbb{R}_+^{n+1} = \{X = (x, x_{n+1}); x \in \mathbb{R}^n, x_{n+1} > 0\}, \quad n \geq 1. \]

In this question the Poisson integral plays an essential role. The Laplace operator \( \Delta \) has constant coefficients; hence the Poisson integral, i.e., a normal derivative of the Green function for the half-space, depends on the difference of arguments and consequently in the harmonic case the Poisson integral is a convolution. In the same article [1] the authors considered the boundary behavior of more general convolution integrals.

In the next step the problem appears to investigate the boundary behavior of nonconvolution integrals of the kind

\[ u(X) = \int_{\mathbb{R}^n} k(X, y)f(y)\,dy, \quad X \in \mathbb{R}_+^{n+1}, \]

which in this context it is natural to call the generalized Poisson integrals.

If \( k \) in (1) is the Poisson kernel for the Laplacian, then the following result is known for the function (1) (see [2, Chapters 3 and 7]).

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Theorem A. Let \( f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty \), and let \( B(x, r) \) be a ball in \( \mathbb{R}^n \) with center \( x \) and radius \( r \), \( \omega_n \) the volume of \( B(0, 1) \), and \( (Mf)(x) = \text{sup}_{r>0}(\omega_n r^n)^{-1} \int_{B(x,r)} |f(y)| \, dy \) the maximal function. Then

1°. \( \text{sup}_{0<x_{n+1}<\infty} |u(x)| \leq (Mf)(x) \) \( \forall x \in \mathbb{R}^n \), as before, here \( X = (x, x_{n+1}) \);

2°. \( \lim_{x_{n+1} \to 0} u(x) = f(x) \) for Lebesgue almost every \( x \in \mathbb{R}^n \);

3°. for \( p < \infty \), \( \lim_{x_{n+1} \to 0} \|u(x) - f(x)\| = 0 \).

Here and in the sequel \( \| \cdot \| \) denotes the norm in \( L^p(\mathbb{R}^n) \).

Actually in [2] the generalization of Theorem A, particularly when the Poisson kernel is replaced by any approximate identity, was proved. In addition the statement was proved in [2] that in 1° and 2° the condition \( x_{n+1} \to 0 \) (i.e., \( X \to x \) along the normal to the boundary \( \mathbb{R}^n \) of \( \mathbb{R}^{n+1} \)) may be interchanged to the tending \( X \to x \) as \( X \) belongs to the cone

\[ \Gamma_\alpha(x_0) = \{X \in \mathbb{R}^{n+1} : |x - x_0| < \alpha x_{n+1} \}, \quad \alpha > 0, \quad x_0 \in \mathbb{R}^n. \]

Our aim is to extend Theorem A on the integrals (1) where the kernel \( k(X, y) \) is defined and measurable on the Cartesian product \( \mathbb{R}^{n+1} \times \mathbb{R}^n \). We investigate the boundary behavior of (1) and apply our results to the Dirichlet problem for the Schrödinger operator \(-\Delta + c(X)I\) in the half-space, \( I \) being the identity operator. Now we state our results. All proofs will be given in §2.

**Proposition 1.** Fix a point \( x \in \mathbb{R}^n \). Suppose the kernel \( k(X, y) \) has the summable majorant \( \psi(x_{n+1}, \cdot) \in L(\mathbb{R}^n) \), depending only on \( |x-y| : |k(X, y)| \leq \psi(x_{n+1}, |x-y|) \) for all \( X = (x, x_{n+1}), 0 < x_{n+1} < h \), with some \( h > 0 \). Let \( \psi(x_{n+1}, r) \) decrease monotonically for \( 0 < r < \infty \) and

\[ A(x_{n+1}) = \int_{\mathbb{R}^n} \psi(x_{n+1}, |y|) \, dy = O(1) \quad \text{as} \quad x_{n+1} \to 0. \]

If \( f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty \), then the following is valid for the function (1):

\[ \limsup_{x_{n+1} \to 0} |u(x)| \leq A(Mf)(x), \]

where \( A = \limsup_{x_{n+1} \to 0} A(x_{n+1}) \).

**Corollary.** If the conditions of the proposition are satisfied with the same \( \psi \) at almost every point \( x \in \mathbb{R}^n \) and \( 1 < p \leq \infty \), then

\[ \limsup_{x_{n+1} \to 0} \|u(\cdot, x_{n+1})\| \leq A\|f\|, \quad A = \text{const}. \]

If \( \sup_{x_{n+1}>0} A(x_{n+1}) < \infty \) and \( p > 1 \), then

\[ \sup_{x_{n+1}>0} \|u(\cdot, x_{n+1})\| \leq A_p\|f\|. \]

**Proposition 2.** Let the conditions of the previous proposition be satisfied for Lebesgue almost every \( x \in \mathbb{R}^n \). Besides let there exist the limit

\[ \lim_{x_{n+1} \to 0} \int_{\mathbb{R}^n} k(X, y) \, dy = 1 \]

for Lebesgue a.e. \( x \in \mathbb{R}^n \) and the limit

\[ \lim_{x_{n+1} \to 0} \int_{|x-y| \geq \delta} \frac{1}{|x-y|^{n-1}} \, dy = 0 \]
for every $\delta > 0$. Then there exists the limit
\[
\lim_{x_{n+1} \to 0} u(X) = f(x) \quad \text{a.e. on } \mathbb{R}^n.
\]

Remark. If in Propositions 1 and 2 the condition $x_{n+1} \to 0$ is replaced by $\Gamma(x_0) \ni X \to x_0$, then all the assertions will remain valid as $\Gamma(x_0) \ni X \to x_0$.

**Proposition 3.** Suppose all conditions of Proposition 2 are satisfied and, in addition, the condition (2) is strengthened to
\[
\lim_{|y| \geq \delta} \int_{\mathbb{R}^n} \psi(x_{n+1}, |y|) \, dy = 0 \quad \forall \delta > 0.
\]
If $1 < p < \infty$ and $|\int_{\mathbb{R}^n} k(X, y) \, dy - 1| \leq \text{const} < \infty$ uniformly in $X \in \mathbb{R}^n \times (0, h)$, then there exists the limit
\[
\lim_{x_{n+1} \to 0} \|u(X) - f(x)\| = 0.
\]

Evidently all our conjectures are fulfilled for the classical Poisson kernel $\gamma_{n+1}x_{n+1}|x - y|^2 + x_{n+1}|x-y|^{(n+1)/2}$, $\gamma_{n+1} = \text{const}$; hence Propositions 1–3 imply Theorem A. Our assumptions are valid too for the kernels
\[
(x_{n+1}^2/(|x - y|^2 + x_{n+1}^2))^{2n/2} \cdot x_{n+1}^{-n}
\]
and
\[
(x_{n+1}^2/(|x - y| + x_{n+1}))^{2n/2} \cdot x_{n+1}^{-n}
\]
with $\lambda > 1$, and Proposition 1 implies the case $z = 0$ and real $\lambda > 1$ of Lemma 3.3 by Johnson [3].

Now let us consider the operator $L_c = -\Delta + c(x)I$, where the function (potential) $c(x) \geq 0$ in $\mathbb{R}^{n+1}_+$ and such that $c \in L^s$ in some neighbourhood of each finite point $X \in \mathbb{R}^{n+1}_+ \cup \mathbb{R}^n$ with certain $s > (n+1)/2$ for $n \geq 3$ and $s = 2$ for $n = 1, 2$. Besides we suppose that $c(x)$ has summable majorant depending only on $|X - y|$ in some vicinity of any boundary point $y \in \mathbb{R}^n$. Under these assumptions it is known that the operator $L_c$ on $L^2(\mathbb{R}^{n+1}_+)$ has a Green function $G(X, Y)$ in $\mathbb{R}^{n+1}_+$ with analytic properties necessary in the sequel. Consequently Propositions 1–3 and the results of [4, 5] imply the following.

**Theorem.** Suppose $f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$. Then the Dirichlet problem
\[
\begin{cases}
-\Delta u + c(X)u, & X \in \mathbb{R}^{n+1}_+, \\
u(x) = f(x) & \text{a.e. on } \mathbb{R}^n
\end{cases}
\]
has the solution
\[
u(X) = \int_{\mathbb{R}^n} \frac{\partial G(X, y)}{\partial n(y)} f(y) \, dy, \quad X \in \mathbb{R}^{n+1}_+,
\]
which satisfies all the conclusions of Propositions 1–3. Here $\partial / \partial n$ is a derivative along an inner normal to $\mathbb{R}^n$. Moreover, this solution $u$ is Hölder continuous with exponent $2 - n/s$ if $n/2 < s \leq n$ and its gradient $\nabla u$ is Hölder continuous with exponent $1 - n/s$ if $s > n$ in all points $X \in \mathbb{R}^{n+1}_+$. For potentials under consideration this assertion makes more precise Simon’s results about Hölder
continuity of the solutions to the Schrödinger equation $L_c u = 0$ [6, Theorems C.2.4 and C.2.5, p. 497].

Of course this solution of problem (3) is not unique. The uniqueness is valid only with a priori growth estimates of the solution at infinity; this question will be considered elsewhere.

2. Demonstrations

We essentially use some ideas of Stein's book [2].

Proof of Proposition 1. By assumption

$$I \equiv \int_{\mathbb{R}^n} \psi(x_{n+1}, |y - x|) dy < \infty.$$  

If we introduce spherical coordinates in $\mathbb{R}^n$ with pole $x$, then $I = n\omega_n \int_0^\infty r^{n-1} \psi(x_{n+1}, r) dr$. The monotonicity of $\psi$ implies $r^n \psi(x_{n+1}, r) \to 0$ as $r \to 0$ and $r \to \infty$. Now integration by parts leads to the equality

$$I = \omega_n \int_0^\infty r^n d\{-\psi(x_{n+1}, r)\}.$$  

Denote (see [2])

$$\lambda(r) = \int_{|t - x| = r} f(t) d\sigma(t)$$

and

$$\Lambda(r) = \int_{|x - y| \leq r} |f(y)| dy = \int_0^r t^{n-1}\lambda(t) dt.$$  

Then $\Lambda(r) \leq \omega_n r^n (Mf)(x)$. Hence if $(Mf)(x) < \infty$, then for $0 < x_{n+1} < h$,

$$\lim r \to 0, r \to \infty.$$

Now we can estimate the function $u$:

$$|u(X)| \leq \int_{\mathbb{R}^n} |k(X, y)||f(y)| dy \leq \int_0^\infty \Lambda(r) d\{ -\psi(x_{n+1}, r)\}$$

with regard to (4). From this we have

$$|u(X)| \leq \omega_n (Mf)(x) \int_0^\infty r^n d\{ -\psi(x_{n+1}, r)\},$$

and after back integration by parts the inequality $|u(X)| \leq A(x_{n+1})(Mf)(x)$ follows. To finish the proof it remains to let $x_{n+1} \to 0$. Q.E.D.

Proof of Proposition 2. Transform the difference

$$u(X) - f(x) = \int_{|y - x| < \delta} k(X, y)\{f(y) - f(x)\} dy$$

$$+ \int_{|y - x| \geq \delta} k(X, y)\{f(y) - f(x)\} dy + f(x)\left( \int_{\mathbb{R}^n} k(X, y) dy - 1 \right)$$

$$\equiv I_1 + I_2 + I_3$$

and estimate every term separately.

Let $x$ be a point of the Lebesgue set of the function $f$; then $|f(x)| < \infty$. If $x$ does not belong to other exceptional sets of zero measure (see the statement of the proposition), i.e., $x$ belongs to the set of full measure in $\mathbb{R}^n$, then

$$\lim_{x_{n+1} \to 0} I_3 = 0$$

by virtue of (2). Similarly, $\lim_{x_{n+1} \to 0} f(x) \int_{|y - x| \geq \delta} k(X, y) dy$
= 0. We estimate the integral $I_{21} = \int_{|y-x| \geq \delta} k(X, y)f(y)\,dy$ by the Hölder inequality:

$$|I_{21}| \leq \|f\| \left\{ \int_{|y-x| \geq \delta} |k(X, y)|^q\,dy \right\}^{1/q}, \quad q = p/(p-1) \geq 1.$$  

The inequality $|k(X, y)| < 1$ is valid asymptotically by virtue of the monotonicity and integrability of the majorant $\psi$. Hence $|k(X, y)|^q \leq |k(X, y)|$ asymptotically and

$$|I_{21}| \leq \text{const}\|f\| \left\{ \int_{|y-x| \geq \delta} |k(X, y)|\,dy \right\}^{1/q} \to 0 \quad \text{as } x_{n+1} \to 0.$$

To estimate $I_1$ we introduce the function as in [2]

$$g(y) = \begin{cases} f(y) - f(x) & \text{if } |y - x| < \delta, \\ 0 & \text{if } |y - x| \geq \delta. \end{cases}$$

For every $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon)$ such that $(Mg)(x) < \varepsilon$ because $x$ is the point of the Lebesgue set of $f$. Therefore Proposition 1 implies $|I_1| \leq A(Mg)(x) < A\varepsilon$, where $\varepsilon$ may be arbitrarily small. Q.E.D.

**Proof of Proposition 3.** We have $\|u(X) - f(x)\| \leq \|I_1\| + \|I_2\| + \|I_3\|$. First

$$|I_1| = \left| \int_{|y-x| < \delta} k(X, y)\{f(y + x) - f(x)\}\,dy \right| \leq \int_{|y| \leq \delta} \psi(x_{n+1}, |y|)|f(y + x) - f(x)|\,dy,$$

but $\psi(x_{n+1}, r)$ does not depend on $x$; hence

$$\|I_1\| \leq \int_{|y| < \delta} \psi(x_{n+1}, r)\|f(y + \cdot) - f(\cdot)\|\,dy.$$  

It is known that the relation $\Delta(y) \equiv \|f(y + \cdot) - f(\cdot)\| = o(1)$ as $|y| < \delta, \delta \to 0$. Hence for arbitrary fixed $\varepsilon > 0$ one can choose $\delta > 0$ such that $\Delta(y) < \varepsilon$ and consequently

$$\|I_1\| \leq \varepsilon \int_{|y| < \delta} \psi(x_{n+1}, |y|)\,dy \leq A(x_{n+1})\varepsilon \leq A\varepsilon.$$  

Now we fix $\delta > 0$ and estimate

$$\|I_2\| \leq \left\| \int_{|y-x| \geq \delta} \psi(x_{n+1}, |y-x|)|f(y) - f(x)|\,dy \right\| \leq \int_{|y| \geq \delta} \psi(x_{n+1}, |y|)\|f(y + \cdot) - f(\cdot)\|\,dy \leq 2\|f\| \int_{|y| \geq \delta} \psi(x_{n+1}, |y|)\,dy = o(1), \quad x_{n+1} \to 0.$$  

To estimate $\|I_3\|$ (now $p < \infty$) we use the representation (see [2]) $f = f_1 + f_2$, where $f_1$ is a continuous function with compact support and $\|f_2\| < \varepsilon$. Hence $\sup|f_1| < \infty$ and the function with compact support

$$\left\{ |f_1(x)| \left( \int_{R^n} k(X, y)\,dy - 1 \right) \right\}^p$$
has a summable majorant. Now from the Lebesgue theorem about dominated convergence we have

$$\lim_{x_{n+1} \to 0} \int_{\mathbb{R}^n} |f_1(x)|^p \left| \int_{\mathbb{R}^n} k(X, y) \, dy - 1 \right|^p \, dx = 0.$$ 

Finally,

$$\left\| f_2(x) \left( \int_{\mathbb{R}^n} k(X, y) \, dy - 1 \right) \right\| \leq \text{const} \| f_2 \| \leq \text{const} \varepsilon. \quad \text{Q.E.D.}$$

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REFERENCES


