NONREGULAR EXTREME POINTS IN THE SET OF MINKOWSKI ADDITIVE SELECTIONS

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Abstract. A function \( s: \mathcal{K}^n \to \mathbb{R}^n \), defined on the family \( \mathcal{K}^n \) of all compact convex and nonempty sets in \( \mathbb{R}^n \), is called a Minkowski additive selection, provided \( s(A + B) = s(A) + s(B) \) and \( s(A) \in A \), whenever \( A, B \in \mathcal{K}^n \). We confirm the conjecture [6] that there exist extremal selections which are not regular (\( s \) is regular if \( s(A) \in \text{ext}\ A \), \( A \in \mathcal{K}^n \)).

Let \( \mathcal{K}^n \) denote the family of all convex compact nonempty subsets of \( \mathbb{R}^n \). A mapping \( T: \mathcal{K}^n \to \mathbb{R}^n \) is called Minkowski additive, or simply additive, if \( T(A + B) = T(A) + T(B) \) for all \( A, B \in \mathcal{K}^n \). Let \( \mathcal{L}^n \) be the vector space of all additive mappings equipped with the weakest topology under which all evaluations \( \mathcal{L}^n \ni T \to T(A) \), \( A \in \mathcal{K}^n \), are continuous. It can be easily seen that the set \( \mathcal{S}^n \subseteq \mathcal{L}^n \) of all selections, i.e., the mappings having the property \( T(A) \in A \), is convex and compact. Let \( \mathcal{E}^n \) be the set of all extreme points of \( \mathcal{S}^n \). An element \( s \in \mathcal{S}^n \) is called regular if \( s(A) \in \text{ext}\ A \), where \( \text{ext}\ A \) denotes the set of all extremal points of \( A \). The set of all regular selections will be denoted by \( \mathcal{R}^n \). Obviously, if \( s \in \mathcal{R}^n \), then \( s \in \mathcal{E}^n \).

Zivaljević [6] conjectured that there exist nonregular extremal points in \( \mathcal{S}^n \). The following result confirms this supposition.

Theorem. For every \( n \geq 2 \), there exists a closed face of \( \mathcal{S}^n \) disjoint with \( \mathcal{R}^n \).

To prove the theorem, we shall need some additional notions and definitions. Let us define the support function \( h(A, x) \) of \( A \in \mathcal{K}^n \) at \( x \) as follows:

\[
h(A, x) = \sup \{ \langle a, x \rangle : a \in A \},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the scalar product. By \( \Omega_k \) we denote the Stiefel manifold of \( k \)-frames; that is, the ordered \( k \)-tuples \( \omega = (x_1, \ldots, x_k) \) of orthonormal vectors in \( \mathbb{R}^n \). For \( \omega \in \Omega_k \), we define the \( \omega \)-face \( V_\omega(A) \) of \( A \) inductively: Let \( V_{x_1}(A) = \{ a \in A : \langle a, x_1 \rangle = h(A, x_1) \} \) and suppose that we have already defined \( V_{\omega'} \) for \( \omega' = (x_1, \ldots, x_{k-1}) \). Then \( V_\omega(A) = V_{x_1}(V_{\omega'}(A)) \). Subsequently, for \( V_{\omega'}(A) \), we define its position vector \( H_{\omega'}(A) \) as follows:

\[
H_{x_1}(A) = h(A, x_1)x_1, \quad H_{\omega'}(A) = H_{\omega'}(A) + h(V_{\omega'}(A), x_k)x_k.
\]

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It was the main result of [6] that \( \mathcal{R}^n = \{ H_\omega : \omega \in \Omega_n \} \). Basic properties of the face mappings can be found in [2, 5].

Suppose now that \( A \in \mathcal{R}^2 \). Let us denote by \( s_0(A) \) the center of the smallest rectangle containing \( A \), which has its sides parallel to the coordinate axes. It is easy to observe that \( s_0(A) \in A \). Moreover, \( s_0(A) \) can be expressed by the formula

\[
s_0(A) = \frac{1}{2}(h(A, e_1) - h(A, -e_1))e_1 + \frac{1}{2}(h(A, e_2) - h(A, -e_2))e_2
\]

where \( e_1, e_2 \) denote vectors of the standard basis in \( \mathbb{R}^2 \). Obviously, \( s_0 \) is an additive selection on \( \mathcal{R}^2 \). This selection has already been mentioned in [1, 4].

**Proposition.** The minimal closed face of \( \mathcal{R}^2 \) containing \( s_0 \) is disjoint with \( \mathcal{R}^2 \).

**Proof.** For every pair \( e_1, e_2 \in \{-1, 1\} \), we define the triangle \( T(e_1, e_2) = \text{conv}\{0, e_1e_1, e_2e_2\} \). It is clear that \( s_0(T(e_1, e_2)) = (e_1e_1 + e_2e_2)/2 \). Hence for every selection \( s \) belonging to the minimal closed face containing \( s_0 \) we have

\[
(*) \quad s(T(e_1, e_2)) \in [e_1e_1, e_2e_2].
\]

On the other hand, it can be easily seen that no regular point \( H_\omega, \omega \in \Omega_2 \), can satisfy all the relations (*) resulting when \( e_1 \) and \( e_2 \) run over \( \{-1, 1\} \). \( \square \)

**Proof of the theorem.** Let us regard \( \mathbb{R}^2 \) as embedded into \( \mathbb{R}^n \) in an obvious manner. Let \( \omega \in \Omega_{n-2} \) consist of elements orthogonal to \( \mathbb{R}^2 \). It is clear that \( V_\omega(A) - H_\omega(A) \in \mathbb{R}^2 \) for every \( A \in \mathcal{R}^n \). Consequently, the following mapping is well defined:

\[
s_\omega(A) = s_0(V_\omega(A) - H_\omega(A)) + H_\omega(A).
\]

It is an easy exercise to prove that \( s_\omega \) is an additive selection on \( \mathcal{R}^n \). Furthermore, it follows from the proposition that the minimal face containing \( s_\omega \) is disjoint with \( \mathcal{R}^n \). \( \square \)

**Question.** Is it true that \( s_\omega \in \mathcal{E}^n \)?

For further information on the topic of extremal selections the reader is referred to the more extensive study [3].

**References**