STRONGLY EXTREME POINTS
AND THE RADON-NIKODÝM PROPERTY

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Abstract. We prove that if \( K \) is a bounded and convex subset of a Banach space \( X \) and \( x \) is a point in \( K \), then \( x \) is a strongly extreme point of \( K \) if and only if \( x \) is a strongly extreme point of \( \bar{K}^* \) which is the weak* closure of \( K \) in \( X^{**} \). We also prove that a Banach space \( X \) has the Radon-Nikodym property if and only if for any equivalent norm on \( X \), the unit ball has a strongly extreme point.

Suppose \( K \) is a subset of a Banach space \( X \) and \( x \in K \). The element \( x \) is called an extreme point of \( K \) if \( x \notin \text{co}(K \setminus \{x\}) \), where \( \text{co}(K \setminus \{x\}) \) is the convex hull of the set \( K \setminus \{x\} \). Various kinds of extreme points have been introduced and studied, among them are denting points and strongly extreme points. Denting points can be defined in terms of slices of \( K \) which are of the form

\[
S(x^*, K, \delta) = \{x \in K : x^*(x) > \sup x^*(K) - \delta\},
\]

where \( \delta \) is a positive number and \( x^* \) is an element in \( X^* \), the dual of \( X \). The element \( x \) is called a denting point of \( K \) if the family of all slices of \( K \) containing \( x \) is a neighborhood base of \( x \) with respect to the relative norm topology on \( K \). It is called a strongly extreme point of \( K \) if for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for any \( y \) in \( X \) the conditions \( d(x + y, K) < \delta \) and \( d(x - y, K) < \delta \) imply that \( ||y|| < \varepsilon \), where \( d(x, K) \) is the distance between \( x \) and \( K \). We use \( \text{ext}\ K \) (resp. \( \text{str-ext}\ K, \text{dent}\ K \)) to denote the set of the extreme (resp. strongly extreme, denting) points of \( K \). It is obvious that if \( x \in \text{dent}\ K \), then \( x \in \text{str-ext}\ K \). In addition, it is easy to see that if \( K \) is convex and \( x \in \text{str-ext}\ K \), then \( x \in \text{ext}\ K \). Let \( \bar{K}^* \) be the weak* closure of \( K \) in \( X^{**} \). An extreme point of \( K \) may not be an extreme point of \( \bar{K}^* \), even if \( K \) is the unit ball of \( X \) \([5]\). On the other hand, we will show that if \( K \) is bounded and convex and \( x \in K \), then \( x \in \text{str-ext}\ K \) if and only if \( x \in \text{str-ext}\ \bar{K}^* \) (see Theorem 3).

Two important properties of Banach spaces, namely, the Radon-Nikodým property (RNP) and the Krein-Milman property (KMP), can be defined in terms of denting points and extreme points respectively. The Banach space \( X \) is said

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to have the RNP (resp. KMP) if every nonempty bounded closed convex set $K$ in $X$ has a denting (resp. extreme) point [1]. It is unknown whether the RNP and the KMP are equivalent. However, using a result of Huff and Morris [3], it can be proved that $X$ has the RNP if and only if every nonempty bounded closed convex set $K$ in $X$ has an extreme point of $K^*$ [1, Corollary 3.76; 4, Remarks, p. 174]. Morris [5] proved that every separable Banach space that contains an isomorphic copy of $c_0$ admits an equivalent strictly convex norm $\| \|$ such that the unit ball $B(x, \| \|)$ of $X$ has no extreme points of the unit ball $B(x, \| \|)$ of $X^{**}$. On the other hand, it is known that $X$ has the RNP if and only if for any equivalent norm on $X$ the respective unit ball $B_x$ has a denting point (see, e.g., [1, p. 30]). Thus, as observed by Morris [5], if $X$ has the RNP, then for any equivalent norm on $X$ the respective unit ball $B_x$ has an extreme point of $B_{x^{**}}$. Morris conjectured [5] that the converse is also true. Though we are not able to prove the conjecture in this paper, we will show that $X$ has the RNP, if and only if for any equivalent norm on $X$ the respective unit ball $B_x$ has a strongly extreme point (see Corollary 6).

For our discussion, we will need several equivalent formulations of strongly extreme points listed in Lemma 1. We omit the proof of Lemma 1 because it is straightforward.

**Lemma 1.** Suppose $K$ is a subset of a Banach space $X$ and $x \in K$. The following are equivalent:

1. $x \in \text{str-ext } K$.
2. For any sequence $\{x_n\}$ in $X$, if $\lim_n d(x + x_n, K) = 0$, then $\lim_n x_n = 0$.
3. For any sequences $\{x_n\}$ and $\{y_n\}$ in $K$, if $\lim_n (x_n + y_n)/2 = x$, then $\lim_n x_n = \lim_n y_n = x$.
4. For any $\varepsilon > 0$ there is a $\delta > 0$, such that for any two vectors $x_1$ and $x_2$ in $K$, if $\|(x_1 + x_2)/2 - x\| < \delta$ then $\|x_1 - x_2\| < \varepsilon$.

Lemma 2 may be used to reduce some problems about general convex sets to problems about symmetric convex sets (see the proof of Theorem 3).

**Lemma 2.** Suppose $K$ is a subset of a Banach space $X$. Let $\text{Sy}(X, K)$ be the convex hull of $\{(x, 1), (-x, -1) : x \in K\}$ and let $\text{Sy}^*(X, K)$ be the weak* closure of $\text{Sy}(X, K)$ in the bidual of the direct sum $X \oplus \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

1. The set $\text{Sy}(X, K)$ is symmetric.
2. If $K$ is bounded, then $\text{Sy}(X, K)$ is bounded and $\text{Sy}^*(X, K) = \text{Sy}(X^{**}, \text{co}^* K)$, where $\text{co}^* K$ is the weak* closure of $\text{co} K$ in $X^{**}$.
3. If $K$ is bounded and convex, then $\text{str-ext } \text{Sy}(X, C) = \{(x, 1), (-x, -1) : x \in \text{str-ext } K\}$.

**Proof.** (1) and (2) are obvious. Without loss of generality, we assume the norm on $X \oplus \mathbb{R}$ is defined by $\|(x, r)\| = \max\{\|x\|, |r|\}$ for every $(x, r)$ in $X \oplus \mathbb{R}$. Let $A = \{(x, 1) : x \in K\}$, and let $B = -A$. Since $\text{Sy}(X, K) = \text{co}(A \cup B)$, we have $\text{str-ext } \text{Sy}(X, K) \subset A \cup B$. Thus $\text{str-ext } \text{Sy}(X, K) \subset \text{str-ext } A \cup \text{str-ext } B$. It is obvious that $\text{str-ext } A = \{(x, 1) : x \in \text{str-ext } K\}$ and $\text{str-ext } B = \{(-x, -1) : x \in \text{str-ext } K\}$. Let $M = \sup\{\|z\| : z \in \text{Sy}(X, K)\}$, and let $x \in K$. Note that $M \geq 1$. If $(x, 1) \notin \text{str-ext } \text{Sy}(X, K)$, then there is $\varepsilon > 0$ such that for any
\( \epsilon/2 > \delta > 0 \) there are \( u_1 \) and \( u_2 \) in \( Sy(X, K) \) satisfying \[
\|(u_1 + u_2)/2 - (x_1, 1)\| < \delta/(6M) \quad \text{and} \quad \|u_1 - u_2\| > \epsilon.
\]

For \( i = 1 \) or \( 2 \), there are \( x_i \) and \( y_i \) in \( K \) and \( t_i \) in \([0, 1]\) such that \( u_i = t_i(x_i, 1) + (1 - t_i)(-y_i, -1) \). It follows that \( 2 - t_1 - t_2 \leq \|(u_1 + u_2)/2 - (x, 1)\| < \delta/(6M) \). Thus \[
\|u_i - (x_i, 1)\| = (1 - t_i)||x_i + y_i, 2|| < \delta/3.
\]

Hence\[
\|x_1 - x_2\| = ||(x_1, 1) - (x_2, 1)|| \geq \|u_1 - u_2\| - \|u_1 - (x_1, 1)\| - \|u_2 - (x_2, 1)\|
\]
\[
> \epsilon - \delta/3 - \delta/3 > \epsilon/2
\]
and\[
\||(x_1 + x_2)/2 - x|| = \||(x_1, 1) + (x_2, 1)||/2 - (x, 1)\|
\]
\[
\leq ||u_1 + u_2)/2 - (x, 1)|| + ||u_1 - (x_1, 1)||/2 + ||u_2 - (x_2, 1)||/2
\]
\[
< \delta/(6M) + \delta/6 + \delta/6 < \delta.
\]

Therefore, \( x \notin \text{str-ext } K \). Similarly if \( (-x, -1) \notin \text{str-ext } Sy(X, K) \), then \( x \notin \text{str-ext } K \). Hence \( \text{str-ext } Sy(X, K) = \{(x, 1), (-x, -1) : x \in \text{str-ext } K \} \). Q.E.D.

**Theorem 3.** If \( K \subset X \) is bounded and convex and \( x \in K \), then \( x \in \text{str-ext } K \) if and only if \( x \in \text{str-ext } \overline{K}^* \).

**Proof.** Since \( K \) is a subset of \( \overline{K}^* \), if \( x \in \text{str-ext } \overline{K}^* \) then \( x \in \text{str-ext } K \). Now suppose \( x \in \text{str-ext } K \). By Lemma 2, we have \( (x, 1) \in \text{str-ext } Sy(X, K) \) and \( Sy^*(X, K) = Sy(X**, \overline{K}^*) \), and \( (x, 1) \in \text{str-ext } Sy(X**, \overline{K}^*) \) if and only if \( x \in \text{str-ext } \overline{K}^* \). Passing to \( (x, 1) \) and \( Sy(X, K) \) if necessary, we may assume that \( K \) is also symmetric. Assume that \( x \notin \text{str-ext } \overline{K}^* \). Then there are \( \epsilon > 0 \) and a sequence \( \{x^n\} \) in \( X** \) such that \( \|x^n\| > \epsilon \) and \( d(x \pm x^n, \overline{K}^*) < 1/n \). For each \( n \geq 1 \), choose \( x^n \in S_{x^n} \) such that \( x^n*(x^n) > \epsilon \), and let \( \|\|_n \) be the Minkowski functional determined by \( K + 1/nB_x \). It is obvious that \( B(x, \|\|_n) = \overline{K}^* + 1/nB_{x**} \). Thus \( \|x \pm x^n\|| < 1 \). By the local reflexivity of Banach spaces [2], for each \( n \geq 1 \) there is a linear operator \( T_n \) from \( \text{span}\{x, x^n\} \) to \( X \) such that
\[
\|T_n(x \pm x^n)\|n < 1, \quad T_n(x) = x, \quad \text{and} \quad x^n*(T_n(x^n*)) = x^n*(x^n*).
\]
Let \( x_n = T_n(x^n*) \). Then \( \|x_n\| \geq x^n*(x_n) = x^n*(x^n*) > \epsilon \) and \( \|x \pm x_n\|| < 1 \). So \( x \pm x_n \in \overline{K} + 1/nB_x \), that is, we have \( d(x \pm x_n, K) \leq 1/n \). Therefore, \( x \notin \text{str-ext } K \), which is a contradiction. Hence \( x \in \text{str-ext } \overline{K}^* \). Q.E.D.

Without assuming \( K \) to be bounded, one can prove that if \( x \in \text{str-ext } K \) then \( x \) is an extreme point of \( \text{ext } \overline{K}^* \) (see [4, Remarks, p. 174] or Lemma 4).

**Lemma 4.** Suppose \( K \subset X \) is convex and \( x \in K \). Consider the following statements:

1. For any sequences \( \{x_n\} \) and \( \{y_n\} \) in \( K \), if \( \lim n(x_n + y_n)/2 = x \), then \( \text{weak-lim } x_n = \text{weak-lim } y_n = x \).
2. For any nets \( \{x_i\} \) and \( \{y_i\} \) in \( K \), if \( \text{weak-lim } x_i + y_i)/2 = x \), then \( \text{weak-lim } x_i = \text{weak-lim } y_i = x \).
(3) The element $x$ is an extreme point of $K^*$.

(4) For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in $K$, if $\lim_n (x_n + y_n)/2 = x$, then weak-$\lim_n x_n = weak-$\lim_n y_n = x$.

(5) For any bounded nets $\{x_\lambda\}$ and $\{y_\lambda\}$ in $K$, if weak-$\lim_\lambda (x_\lambda + y_\lambda)/2 = x$, then weak-$\lim_\lambda x_\lambda = weak-$\lim_\lambda y_\lambda = x$.

Then (1) and (2) are equivalent and each of them implies (3). Statements (4) and (5) are equivalent and both are implied by (3). Thus if, in addition, the set $K$ is bounded, then all the above statements are equivalent.

Proof. It is obvious that (2) implies (1). To prove the converse is true, we assume that there exist some nets $\{x_\lambda\}$ and $\{y_\lambda\}$ in $K$ such that weak-$\lim_\lambda (x_\lambda + y_\lambda)/2 = x$, but $\{x_\lambda\}$ does not converge weakly to $x$. Passing to subnets if necessary, we may assume that there is an extreme point $x^*$ in $X^*$ such that $x^*(x_\lambda - x) > 1$. Since weak-$\lim_\lambda (x_\lambda + y_\lambda)/2 = x$, we may assume that $x^*(x_\lambda - y_\lambda) > 0$. Let $A = co\{x_\lambda\}$ and $B = co\{y_\lambda\}$. Then $\inf x^*(A) > \max x^*(B) + 1$ and there is a sequence $z_n$ in $co\{(x_\lambda + y_\lambda)/2\}$ such that $\lim_n z_n = x$. Hence there are sequences $\{x_n\}$ in $co\{x_\lambda\}$ and $\{y_n\}$ in $co\{y_\lambda\}$ such that $(x_n + y_n)/2 = z_n$. Thus $\lim_n (x_n + y_n)/2 = x$ and $x^*(x_n - y_n) > 1$, which implies that either $\{x_n\}$ or $\{y_n\}$ does not converge weakly to $x$. Therefore, (1) implies (2).

The proof of the equivalence of (4) and (5) is similar.

Assume that $x$ is not an extreme point of $K^*$. Then there are $x^{**}$ and $y^{**}$ in $K^*$ such that $x^{**} \neq x \neq y^{**}$ and $x = (x^{**} + y^{**})/2$. Choose $x^*$ in $X^*$ such that $(x^{**} - x)(x^*) > 1$. Then $(x - y^{**})(x^*) > 1$. There exist nets $\{x_\lambda\}$ and $\{y_\lambda\}$ in $K$ such that weak-$\lim_\lambda x_\lambda = x^{**}$ and weak-$\lim_\lambda y_\lambda = y^{**}$. Thus weak-$\lim_\lambda (x_\lambda + y_\lambda)/2 = x$, but $\{x_\lambda\}$ does not converge weakly to $x$. Hence (2) implies (3).

Finally, to show (3) implies (5), we assume that $x$ is an extreme point of $K^*$. Suppose $\{x_\lambda\}$, $\{y_\lambda\}$ are two bounded nets in $K$ with weak-$\lim_\lambda (x_\lambda + y_\lambda)/2 = x$. Then $\{x_\lambda\}$ has a weak* cluster point. Let $x^{**}$ be a weak* cluster point of $\{x_\lambda\}$. Then $x^{**} \in K^*$ and there is a subnet $\{x_{\lambda(\alpha)}\}$ of $\{x_\lambda\}$ such that weak-$\lim_{\lambda(\alpha)} x_{\lambda(\alpha)} = x^{**}$. Since weak-$\lim_\lambda (x_\lambda + y_\lambda)/2 = x$, the weak* limit of $\{y_{\lambda(\alpha)}\}$ exists, say, weak-$\lim_{\lambda(\alpha)} y_{\lambda(\alpha)} = y^{**}$. Then $y^{**} \in K^*$ and $x = (x^{**} + y^{**})/2$. Since $x$ is an extreme point of $K^*$, we can conclude that $x^{**} = x$. Hence weak-$\lim_\lambda x_\lambda = weak-$\lim_\lambda y_\lambda = x$. Q.E.D.

Theorem 5. Suppose $K_1$, $K_2 \subset X$ are closed and convex, and one of them is bounded and $x \in X$. Let $K$ be the weak* closure of $K_1 + K_2$ in $X^{**}$. If $x$ is an extreme point of the weak* closure of $K$ in $X^{(4)}$, the fourth dual of $X$, then $x$ is in $K_1 + K_2$. In particular, if $x$ is a strongly extreme point of the norm closure of $K_1 + K_2$, then $x$ is in $K_1 + K_2$.

Proof. It is obvious that the weak* closure of $K_1 + K_2$ is $\overline{K_1}^* + \overline{K_2}^*$. Thus there are $u_1$ in $\overline{K_1}^*$ and $u_2$ in $\overline{K_2}^*$ such that $x = u_1 + u_2$. We can choose sequences $\{x_1(n)\}$ in $K_1$ and $\{x_2(n)\}$ in $K_2$ such that $\lim_n x_1(n) + x_2(n) = x$. Let $y_1(n) = x_1(n) + u_2$ and $y_2(n) = x_2(n) + u_1$. Then $\{y_1(n)\}$ and $\{y_2(n)\}$ are bounded sequences in $\overline{K_1}^* + \overline{K_2}^*$. Since $\lim_n [y_1(n) + y_2(n)]/2 = x$, we have weak-$\lim_n y_1(n) = weak-$\lim_n y_2(n) = x$. Thus by Lemma 4 the sequence $\{x_i(n)\}$ is weakly convergent for $i = 1$ and 2. It follows that $u_i \in K_i$. Therefore, $x$ is in $K_1 + K_2$. Now suppose $x$ is a strongly extreme point of
The norm closure of $K_1 + K_2$. By Theorem 3, the element $x$ is also a strongly
extreme point of $K$. Thus $x$ is an extreme point of the weak* closure of $K$
in $X^{(4)}$. Therefore, $x$ is in $K_1 + K_2$. Q.E.D.

**Corollary 6.** Let $X$ be a Banach space. The following are equivalent:

1. The space $X$ has the RNP.
2. For any equivalent norm $\| \|$ on $X$, the unit ball $B_{(X,\| \|)}$ has a strongly
   extreme point.
3. For any equivalent norm $\| \|$ on $X$, the unit ball $B_{(X,\| \|)}$ has an extreme
   point of $B_{(X^{(4)},\| \|)}$.

**Proof.** It is obvious that (1) implies (2). By Theorem 3, every strongly extreme
point of $B_X$ is an extreme point of $B_{X^{(4)}}$. Thus (2) implies (3). So it remains
to show that (3) implies (1). Let $A \subset X$ be nonempty, bounded, and weakly
closed. Let $K = \overline{co}(A \cup -A)$, and let $\| \|$ be the Minkowski functional
determined by $K + B_X$ where $B_X$ is the unit ball of $X$ with respect to the original norm.
Then $\| \|$ is an equivalent norm on $X$ such that the unit ball $B_{(X,\| \|)}$ is the
norm closure of $K + B_X$. Let $x$ be an element in $X$ such that $x$ is an extreme
point of the unit ball $B_{(X^{(4)},\| \|)}$ of $X^{(4)}$. By Theorem 5, there are $y \in K$ and
$z \in B_X$ such that $x = y + z$. It is obvious that $x$ is an extreme point of $B_{(X^{\infty},\| \|)}$ and $B_{(X^{\infty},\| \|)} = \overline{K}^* + B_{X^{\infty}}$. Thus $y$ is an extreme point of $\overline{K}^*$. By
the Krein-Milman Theorem, the set $\text{ext} \overline{K}^*$ is contained in the weak* closure
of $A \cup -A$. Since $A$ is weakly closed, we have $y \in A$ or $y \in -A$. In any
case the set $A$ has an extreme point. Therefore, $X$ has the RNP [1, Corollary
3.76]. Q.E.D.

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