TYPICAL INTERSECTIONS OF CONTINUOUS FUNCTIONS WITH MONOTONE FUNCTIONS

M. HEJNY

(Communicated by Andrew M. Bruckner)

Abstract. For each parameter $\Phi$ a typical continuous function intersects every monotone function in a $(\Phi)$-uniformly symmetrically porous set.

1. Introduction and notation

This paper generalizes the results of Humke and Laczkovich [1]. The notion of "bilaterally strongly $\Phi$-porosity" in [1] is replaced by the stronger one of "$(\Phi)$-uniformly symmetric porosity" (see Definition 2 and Theorem 3). The main result of the paper, Theorem 4, is obtained using an adaptation of the Banach-Mazur game (Theorem 6), and there is a different approach than that used in [1].

Definition 1. Let $\Phi: (0, 1) \rightarrow (0, 1]$ be a continuous function. A set $E \subset \mathbb{R}$ is said to be bilaterally strongly $\Phi$-porous if for every $x \in E$ there are sequences of intervals $I_n \subset (x - 1/n, x) \setminus E$ and $J_n \subset (x, x + 1/n) \setminus E$ such that

$$\lim_{x \to \pm \infty} \frac{\text{dist}(x, I_n)}{\Phi(|I_n|)} = \lim_{x \to \pm \infty} \frac{\text{dist}(x, J_n)}{\Phi(|J_n|)} = 0.$$

In [1] Humke and Laczkovich proved the following

Theorem 1. Let $\Phi: (0, 1) \rightarrow (0, 1]$ be a continuous function. Then a typical continuous function intersects every monotone function in a bilaterally strongly $\Phi$-porous set.

Notation 1. The family of all continuous increasing functions $\Phi$ on $[0, 1]$ for which $\Phi(0) = 0$ will be denoted by $G$; such functions will be referred to as porosity indices.

Notation 2. Let $\Phi \in G$ and $k \in \mathbb{N}$. By $R(\Phi, k)$ we will denote the set of all $E \subset \mathbb{R}$ for which there are numbers $a_k, b_k$ such that for all $x \in E$ the following hold:

(i) $0 < a_k < b_k < k^{-1}$,
(ii) $\Phi(b_k - a_k) > a_k$, and
(iii) $[x - b_k, x - a_k] \cap E = [x + a_k, x + b_k] \cap E = \emptyset$.

Further let us denote $R(\Phi) = \bigcap_{k=1}^{\infty} R(\Phi, k)$.

**Definition 2.** Let $\Phi \in G$. Those sets $E \in R(\Phi)$ are said to be $(\Phi)$-uniformly symmetrically porous.

For our purposes Theorem 1 will be slightly reformulated.

**Definition 3.** Let $\Psi \in G$. We call a set $E \in \mathbb{R}^n$ $(\Psi)$-bilaterally strongly porous if for every $x \in E$ there are sequences of intervals $I_n \subset (x - 1/n, x) \setminus E$ and $J_n \subset (x, x + 1/n) \setminus E$ such that for every $n \in \mathbb{N}$ both $\text{dist}(x, I_n) < \Psi(|I_n|)$ and $\text{dist}(x, J_n) < \Psi(|J_n|)$.

We reformulate Theorem 1 as

**Theorem 2.** Let $\Psi \in G$. Then the typical continuous function intersects every monotone function in a $(\Psi)$-bilaterally strongly porous set.

In fact, for each continuous function $\Phi : (0, 1] \to (0, 1]$ there exists $\Psi_0 \in G$ such that $\Psi_0(x) \leq \Phi(X)$ for $x \in (0, 1]$. Let us set $\Psi(x) = x\Psi_0(x)$. For $\Psi \in G$ we can find (according to Theorem 2) sequences of intervals $\{I_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, $\text{dist}(x, I_n) < \Psi(|I_n|)$ and $\text{dist}(x, J_n) < \Psi(|J_n|)$. Then $\text{dist}(x, I_n) < \Psi(|I_n|) = |I_n|\Psi_0(|I_n|) \leq n^{-1}\Phi(|I_n|)$. Similarly $n \text{dist}(x, J_n) < \Phi(|J_n|)$ and, hence, (a).

Therefore Theorem 1 is a consequence of Theorem 2. The fact that Theorem 2 is a consequence of Theorem 1 is evident.

### 2. Proof of Theorem 3

In this section we will show by Theorem 3 that our main result (Theorem 4) is stronger than the result of Humke and Laczkovich (Theorem 2).

**Theorem 3.** Let $\Phi$ be the identity function on $[0, 1]$. Then for every $\Psi \in G$ there is $E \subset [0, 1]$ such that $E$ is $(\Psi)$-bilaterally strongly porous and is not $(\Phi)$-uniformly symmetrically porous.

**Proof.** Let $\Psi \in G$. The subsequence $\{p_{k_n}\}_{n=1}^{\infty}$ of the sequence $\{p_k\}_{k=1}^{\infty} = \{(\frac{2}{3})^k\}_{k=1}^{\infty}$ will be defined by induction; set $k_1 = 1$ and suppose $k_1, k_2, \ldots, k_n$ are given. Since $\Psi \in G$, there exists $k_{n+1} \in \mathbb{N}$ such that $\Psi(p_{k_{n+1}} - p_{k_n}) > p_{k_{n+1}}$.

Now denote

$$E = \{0.5 + p_m : k_{2n-1} \leq m \leq k_{2n}, \, n \in \mathbb{N}\} \cup \{0.5\} \cup \{0.5 - p_m : k_{2n} \leq m \leq k_{2n+1}, \, n \in \mathbb{N}\}.$$

Obviously there is only one point, namely, $0.5$, for which it is necessary to verify $(\Psi)$-bilaterally strongly porosity. Setting

$$I_n = (0.5 - p_{k_{2n-1}}, \, 0.5 - p_{k_{2n}})$$

and

$$J_n = (0.5 + p_{k_{2n+1}}, \, 0.5 + p_{k_{2n}}),$$

it is easy to see that the set $E$ is $(\Psi)$-bilaterally strongly porous. Further for each $a \in (0, 0.2)$ either $[0.5 - 2a, 0.5 - a] \cap E \neq \emptyset$ or $[0.5 + a, 0.5 + 2a] \cap E \neq \emptyset$. Thus the set $E$ is not $(\Phi)$-uniformly symmetrically porous. Q.E.D.
In Theorem 3 the set $E$ could have been chosen as

$$E = \{0.25 - p_m : k_{2n-1} \leq m \leq k_{2n}, \ n \in \mathbb{N}\} \cup \{0.25 + p_m : k_{2n-1} \leq m \leq k_{2n}, \ n \in \mathbb{N}\} \cup \{0.75 - p_m : k_{2n} < m < k_{2n+1}, \ n \in \mathbb{N}\} \cup \{0.75 + p_m : k_{2n} < m < k_{2n+1}, \ n \in \mathbb{N}\} \cup \{0.25, 0.75\}.$$ 

Now for each $x \in E$ and each $k \in \mathbb{N}$ there are numbers $a_k, b_k$ such that (i), (ii), and (iii) of Notation 1 are fulfilled, but the uniformity is not preserved, i.e., there are no sequences $\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty}$ such that for each $x \in E$ (in our case we put $x = 0.25$ or $x = 0.75$) (i), (ii), and (iii) hold.

The main theorem of this paper reads as follows.

**Theorem 4.** Let $\Phi \in G$. Then the typical continuous function intersects each monotone function in a $\Phi$-uniformly symmetrically porous set.

It is clear that Theorem 2 is a consequence of Theorem 4, but by Theorem 3 we showed that Theorem 2 is not a consequence of Theorem 4.

### 3. Proof of Theorem 2

**Notation 3.** Let $f$ be a real function, $g \in C[0, 1]$, $x \in \mathbb{R}$, $\varepsilon > 0$. We denote

$$U(x, \varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\},
$$

$$U(g, \varepsilon) = \{\phi \in C[0, 1] : |\phi(x) - g(x)| < \varepsilon \text{ for } x \in [0, 1]\},
$$

$$M_{f, \varepsilon} = \{(x, y) \in \mathbb{R}^2 : |f(x) - y| < \varepsilon\},$$

and

$$\text{gr } f = \{(x, f(x)) \in \mathbb{R}^2 : x \in [0, 1]\}.$$ 

For a set $M \subset \mathbb{R}^2$ we denote $P(M) = \{x \in \mathbb{R} : \text{there exists } y \in \mathbb{R} \text{ such that } (x, y) \in M\}.$

We need two lemmas. The first of these is easy to see and is not proved.

**Lemma 1.** Let $\delta > 0$, $\alpha > 0$. Let $r$ be a continuous piecewise linear function on interval $I$, for which $r'_+(x) < -\alpha$, for all $x \in \text{int } I$. Let $f$ be a nondecreasing function. Then there exists an interval of length $2\delta/\alpha$ which contains the set $P(M_{r, \delta} \cap \text{gr } f)$.

**Lemma 2.** Let $U \subset C[0, 1]$ be open and nonempty. Then there exists $n \in \mathbb{N}$ such that for an arbitrary $\gamma > 0$ there is a function $s \in C[0, 1]$ and a number $\delta > 0$ such that for each nondecreasing function $f$ there are intervals $J_1, J_2, \ldots, J_n$ for which

(i) $|J_i| < \gamma$ for $i = 1, 2, \ldots, n$,

(ii) $U(s, \delta) \subset U$, and

(iii) $P(M_{s, \delta} \cap \text{gr } f) \subset \bigcup_{i=1}^{n} J_i$.

**Proof.** Obviously there exists a continuous and piecewise linear function $h \in C[0, 1]$ and a number $\varepsilon > 0$ such that $U(h, \varepsilon) \subset U$. Denote

$$(1) \quad t = \max\{|h'_+(x)| : x \in (0, 1)\}.$$

choose \( n_0 \in \mathbb{N} \) such that

\[
(2) \quad n_0 > \frac{t + 1}{\varepsilon},
\]

and denote \( n = 2n_0 \). Further let \( \gamma > 0 \), \( b = \min \{1/n, \gamma/2\} \), and \( I = [b, 1/n_0] \). Let \( u_0 \) be the continuous and piecewise linear function defined as follows:

(a) \( u_0 \) is periodic with the period \( n_0^{-1} \);
(b) \( u_0(0) = -\frac{\varepsilon}{2} \), \( u_0(b) = \frac{\varepsilon}{2} \), \( u_0(n_0^{-1}) = -\frac{\varepsilon}{2} \); and
(c) \( u_0 \) is linear on \([0, b]\) and on \( I \).

Let us denote \( u(x) = u_0(x)_{|[0, 1]} \), and let \( \delta > 0 \) be such that

\[
(3) \quad 2\delta < \min \{\gamma, \varepsilon\}.
\]

Suppose \( f \) is nondecreasing. Denote \( s = h + u \) and \( J_{i+1} = [in_0^{-1}, in_0^{-1} + b] \) for \( i = 0, 1, \ldots, n_0 - 1 \). The upcoming construction of intervals \( J_{n_0+1}, J_{n_0+1}, \ldots, J_n \) will use Lemma 1.

From (2) and the definition of the function \( u(x) \) it follows that \( u'(x) = -\varepsilon/(n_0^{-1} - b) < -\varepsilon n_0 < -t - 1 \) for \( x \in \text{int} I \). Thus by (1)

\[s'_n(x) = u'_n(x) + h'_n(x) < -t - 1 + t = -1\]

for \( x \in \text{int} I \). Denote \( s_1 = s_{|I} \). From (3) and the fact that \( s'_n(x) < -1 \) for \( x \in \text{Int} I \) it follows from Lemma 1 that there is an interval \( J_{n_0+1} \) of the length less than \( \gamma \) for which \( P(M_{s_1}, \delta \cap \text{gr} f) \subset J_{n_0+1} \). Similarly we define the intervals \( J_{n_0+i} \) for \( i = 2, 3, \ldots, n_0 - 1 \). Thus for \( s_i = s_{|[i(i-1)/n_0+b, 1/n_0]} \) it follows that \( P(M_{s_i}, \delta \cap \text{gr} f) \subset J_{n_0+i} \). Conclusions (i) and (iii) follow directly from the definition of the intervals \( J_1, \ldots, J_n \). From (3) and the inequality \( |u(x)| \leq \varepsilon/2 \) it follows that \( U(s, \delta) \subset U(h, \varepsilon) \); but then \( U(h, \varepsilon) \subset U \) and conclusion (ii) follows. Q.E.D.

To prove Theorem 4 we introduce the Banach-Mazur game: Assume that \( X \) is a complete metric space and \( B \subset X \). The Banach-Mazur game is played by two players, (A) and (B). In the first step, (A) chooses an open and nonempty set \( U_1 \subset X \), and (B) chooses an open and nonempty set \( V_1 \subset U_1 \). In the \( n \)th step (A) chooses an open and nonempty set \( U_n \subset V_{n-1} \) and (B) chooses an open and nonempty set \( V_n \subset U_n \). This defines a nonincreasing sequence of open sets. If \( \bigcap_{i=1}^{\infty} V \subset B \) then (B) wins. In the opposite case (A) wins.

**Theorem 5** (see [1]). In the Banach-Mazur game there is a winning strategy for the player (B) if and only if the set \( B \) is residual in \( X \).

Further the Banach-Mazur game will be looked at with respect to the space \( X = C[0, 1] \).

**Definition 4.** A nonempty family \( P \) of subsets of \( \mathbb{R} \) is called a family of small sets if the relation \( A \in P \) and \( B \subset A \) yields \( B \in P \).

**Notation 4.** Let \( P \) be a family of small sets. The Banach-Mazur game for which \( B = \{\varphi \in C[0, 1]: P(\text{gr} f \cap \text{gr} \varphi) \in P \text{ for every nondecreasing function } f\} \) will be denoted by \( \text{BM}(P) \).

Further we will use the \( F(P) \) game, which is described as follows: The \( F(P) \) game is played by two players, (A) and (B). In the first step, (A) chooses a
number $n_1 \in \mathbb{N}$ and (B) chooses a real positive number $\gamma_1 > 0$. In the $k$th step (A) chooses a number $n_k \in \mathbb{N}$ and (B) chooses a real positive number $\gamma_k > 0$. This defines a sequence $n_1, \gamma_1, n_2, \gamma_2, \ldots$. If for every sequence $\{T_k\}_{k=1}^{\infty}$ of sets $T_k = \bigcup_{i=1}^{n_k} I_i^k$, where $I_i^k$ ($i = 1, 2, \ldots, n_k$) are intervals shorter than $\gamma_k$, $\bigcap_{k=1}^{\infty} T_k \in P$ holds, then (B) wins. In the opposite case (A) wins.

**Lemma 3.** Let $P$ be a family of small sets. If there is a winning strategy for the player (B) in the $F(P)$ game then there is in the $BM(P)$ game as well.

**Proof.** Suppose that the $BM(P)$ game up to the $k$th step of (A) is given by the sequence $U_1 \supset V_1 \supset \cdots \supset U_k$, and suppose that the $F(P)$ game up to the $(k-1)$th step of (B) is given by the sequence $n_1, \gamma_1, \ldots, n_{k-1}, \gamma_{k-1}$ and that (B) has used a winning strategy. Now for $U_k \subset \mathcal{C}[0, 1]$ by Lemma 2 we find $n_k \in \mathbb{N}$. Then with respect to the winning strategy of (B) in the $F(P)$ game for $n_k \in \mathbb{N}$ we obtain a number $\gamma_k > 0$. For $U_k, n_k$, and $\gamma_k$ by Lemma 2 we find a function $s \in \mathcal{C}[0, 1]$ and a number $\delta_k > 0$. Now put $V_k = U(s, \delta_k)$ as the $k$th step of (B). From Lemma 2 we have $V_k \subset U_k$.

We are going to show that (B) wins in this $BM(P)$ game. If $\bigcap_{i=1}^{\infty} V_i = \emptyset$ there is nothing to be proved. Let $s \in \bigcap_{i=1}^{\infty} V_i$ and let $f$ be a nondecreasing function. Denote $E = P(gr f \cap gr s)$. According to Lemma 2 for each $k \in \mathbb{N}$ there are intervals $I_1^k, I_2^k, \ldots, I_n^k$ such that $|I_i^k| < \gamma_k$, $i = 1, 2, \ldots, n_k$, and

$$P(gr f \cap M_{s_k, \delta_k}) \subset T_k \text{ where } T_k = \bigcup_{i=1}^{n_k} I_i^k.$$  

Obviously $E \subset P(gr f \cap M_{s_k, \delta_k})$, and regarding (4) we have

$$E \subset \bigcap_{k=1}^{\infty} T_k.$$  

Since (B) used the winning strategy in the $F(P)$ game, it follows that $\bigcap_{i=1}^{\infty} T_k \in P$. From (5) and the fact that $P$ is a family of small sets we get $E \in P$. Hence (B) wins in the $BM(P)$ game as well. Q.E.D.

**Notation 5.** Let $P$ be a family of small sets. Then we denote

$W_P = \{ \varphi \in \mathcal{C}[0, 1] : P(gr f \cap gr \varphi) \in P \text{ for each } f \text{ monotone} \}$,

$W_P^- = \{ \varphi \in \mathcal{C}[0, 1] : P(gr f \cap gr \varphi) \in P \text{ for each } f \text{ nonincreasing} \}$,

$W_P^+ = \{ \varphi \in \mathcal{C}[0, 1] : P(gr f \cap gr \varphi) \in P \text{ for each } f \text{ nondecreasing} \}$.

**Lemma 4.** Let $P$ be a family of small sets. The set $W_P^+$ is residual if and only if the set $W_P^-$ is residual.

**Proof.** Obviously it is enough to find a homeomorphism $H$ from $\mathcal{C}[0, 1]$ onto $\mathcal{C}[0, 1]$ such that $H(W_P^+) = W_P^-$. Define $H(\varphi) = -\varphi$ for $\varphi \in \mathcal{C}[0, 1]$. Let $\varphi \in W_P^+$, and let $f$ be nonincreasing. Then $P(gr f \cap gr -\varphi) = P(gr -f \cap gr \varphi) \in P$. Thus $-\varphi \in W_P^-$. Similarly $\varphi \in W_P^-$ yields $-\varphi \in W_P^+$. Q.E.D.

**Lemma 5.** Let $P$ be a family of small sets such that the set $W_P^+$ is residual. Then $W_P$ is also residual in $\mathcal{C}[0, 1]$.

**Proof.** According to Lemma 4 the set $W_P = W_P^+ \cap W_P^-$ is residual. Because $W_P = W_P^+ \cap W_P^-$, $W_P$ is thus residual also. Q.E.D.
Theorem 6. Let $P$ be a family of small sets. If there is a winning strategy for the player $(B)$ in the $F(P)$ game then the set $W_P$ is residual in $C[0, 1]$.

Proof. According to Lemma 3 there is a winning strategy for $(B)$ also in the $BM(P)$ game, and with respect to Theorem 5 the set $W^+_P$ is residual. Theorem 6 now follows from Lemma 5. Q.E.D.

Proof of Theorem 4. Let $\Phi \in G$. We need to show that $W_{R(\Phi)}$ is residual. To do that it is enough to verify the assumptions of Theorem 5. Obviously it is enough to find the winning strategy for player $(B)$ in the $F(R(\Phi))$ game. Let the $k$th step of player $(A)$ be given by the number $n_k \in N$. Add We are going to find a number $\gamma_k > 0$ as a $k$th step of player $(B)$. The set $A = \{(x_1, x_2, \ldots, x_{n_k}) \in \mathbb{R}^{n_k} : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_{n_k} \leq 1\}$, as a subset of the metric space $(\mathbb{R}^{n_k}, \rho)$ with maxim metric $\rho$ (i.e., $\rho(x, y) = \max\{|x_i - y_i| : i = 1, 2, \ldots, n_k\}$) where $x = (x_1, x_2, \ldots, x_{n_k})$ and $y = (y_1, y_2, \ldots, y_{n_k})$, is compact. For $x = (x_1, x_2, \ldots, x_{n_k}) \in A$ we define function $F(x) = \sup \{\varepsilon > 0 : \bigcup_{i=1}^{n_k} U(x_i, \varepsilon) \in R(\Phi, k)\}$. Further we will prove that there is a number $\gamma_k > 0$ such that

$$F(x) > \gamma_k \quad \text{for all } x \in A.$$  

Because $A$ is compact, in order to prove (6) it is sufficient to prove the following two assertions:

(7) $F(x) > 0 \quad \text{for all } x \in A,$

and

(8) $F$ is continuous on $A$.

The proof of (7) will be divided into two cases.

I. If $x_i = x_{n_k}$, denote $b_k = 1/(k + 1)$. Because $\Phi \in G$, there exists $a_k > 0$ such that $\Phi(b_k - a_k) > a_k$. For $\varepsilon = 2^{-1}a_k$ obviously $\bigcup_{i=1}^{n_k} U(x_i, \varepsilon) \in R(\Phi, k)$.

II. If $x_i \neq x_{n_k}$, denote $d = \min\{|x_i - x_j| > 0 : i, j = 1, 2, \ldots, n_k\}$ and $b_k = \min\{d/2, 1/(k + 1)\}$. It is clear that there is a number $a_k > 0$ such that $\Phi(b_k - a_k) > a_k$. For $\varepsilon = 2^{-1}a_k$ it is $\bigcup_{i=1}^{n_k} U(x_i, \varepsilon) \in R(\Phi, k)$.

To prove (8) it is sufficient to show

Lemma 6. Let $x = (x_1, x_2, \ldots, x_{n_k}) \in A$, $y = (y_1, y_2, \ldots, y_{n_k}) \in A$, and $\rho(x, y) < \delta$. Then $F(x) + \delta \geq F(y)$.

Proof. If $F(y) \leq \delta$ then clearly Lemma 6 holds. If $F(y) > \delta$ then there exists $\varepsilon > \delta$ such that $\bigcup_{i=1}^{n_k} U(x_i, \varepsilon) \in R(\Phi, k)$. From $\rho(x, y) < \delta$ we have $\bigcup_{i=1}^{n_k} U(x_i, \varepsilon - \delta) \subset \bigcup_{i=1}^{n_k} U(y_i, \varepsilon)$, and since $R(\Phi, k)$ is a family of small sets, we obtain $\bigcup_{i=1}^{n_k} U(x_i, \varepsilon - \delta) \in R(\Phi, k)$. Q.E.D.

To finish the proof of Theorem 4 we will show that in the described game given the $n_1, \gamma_1, n_2, \gamma_2, \ldots$ $(B)$ wins.

Let there be a sequence $\{T_k\}_{k=1}^{\infty}$, $T_k = \bigcup_{i=1}^{n_k} I_i^k$, where intervals $I_i^k$ $(i = 1, 2, \ldots, n_k)$ satisfy $|I_i^k| < \delta_k$. Then there exists $x = (x_1, x_2, \ldots, x_{n_k}) \in A$ such that $T_k \subset \bigcup_{i=1}^{n_k} U(x_i, \gamma_k)$. With respect to (6) and to the fact that $R(\Phi, k)$ is a family of small sets we obtain $T_k \in R(\Phi, k)$. Hence $\bigcap_{i=1}^{\infty} T_i \in R(\Phi, k)$, and also $\bigcap_{i=1}^{\infty} R(\Phi, k) = R(\Phi)$. Q.E.D.

It is not very difficult to show that Theorem 4 yields the following assertion.
Let $\Phi \in G$. Then a typical continuous function intersects every Lipschitz function in $(\Phi)$-uniformly symmetrically porous set.

ACKNOWLEDGMENT

The results in this paper are part of my diploma thesis (Charles University of Prague, 1989) written under the direction of Professor L. Zajíček. I thank him for his help and encouragement.

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DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF MATHEMATICS AND PHYSICS, COMENIUS UNIVERSITY, MLYNSKÁ DOLINA, 842 15 BRATISLAVA, CZECHOSLOVAKIA

E-mail address: hejny@mff.uniba.cs