THE HAUSDORFF DIMENSION OF SELF-SIMILAR SETS UNDER A PINCHING CONDITION

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Abstract. We study self-similar sets in the case where the construction diffeomorphisms are not necessarily conformal. Using topological pressure we give an upper estimate of the Hausdorff dimension, when the construction diffeomorphisms are $C^{1+\kappa}$ and satisfy a $\kappa$-pinching condition for some $\kappa < 1$. Moreover, if the construction diffeomorphisms also satisfy the disjoint open set condition we then give a lower bound for the Hausdorff dimension.

1. Introduction

The construction of a self-similar set starts with a $k \times k$ matrix $A = (a_{ij})$ which has entries zeros and ones, with all entries of $A^N$ positive for some $N > 0$; see [H]. For each nonzero $a_{ij}$ we give a contraction map $\varphi_{ij} : \mathbb{R}^l \rightarrow \mathbb{R}^l$ with $\|\varphi_{ij}(x) - \varphi_{ij}(y)\| \leq c\|x - y\|$, where $c < 1$ is a constant and we are using the Euclidean norm on $\mathbb{R}^l$. Define the Hausdorff metric by

$$d(E, F) = \inf\{\delta \mid d(x, F) \leq \delta \text{ for all } x \in E, \text{ and } d(y, E) \leq \delta \text{ for all } y \in F\}$$

in the space $\mathcal{H}$ of all nonempty compact subsets of $\mathbb{R}^l$. See, for example, [H] or [F]. The map $\Phi$ on the $k$-fold product space $\mathcal{H}^k$ given by

$$\Phi(F_1, \ldots, F_l) = \left( \bigcup_{j=1}^k \varphi_{1j}(F_j), \ldots, \bigcup_{j=1}^k \varphi_{kj}(F_j) \right)$$

is a contraction map. By the Banach Fixed Point Theorem the contraction map $\Phi$ has a unique fixed point in $\mathcal{H}^k$, i.e., a vector of compact nonempty subsets of $\mathbb{R}^l$, $(E_1, \ldots, E_k) \in \mathcal{H}^k$, with $\bigcup_{a_{ij}=1} \varphi_{ij}(E_j) = E_i$. The union $E = \bigcup_{i=1}^k E_i$ is called a self-similar set.

Let $\Sigma = \Sigma^+_A = \{(x_0, x_1, \ldots, x_n, \ldots) \mid 1 \leq x_i \leq k \text{ and } a_{x_i,x_{i+1}} = 1 \text{ for all } i \geq 0\}$ be the shift space with the following metric: for $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots)$ in $\Sigma$, $d(x, y) = 2^{-n}$ if and only if $n = \min\{m \mid x_m \neq y_m\}$. Let $\sigma$ be the shift map of $\Sigma$, and let $\pi : \Sigma \rightarrow E$ be given by

$$\pi(x_0, x_1, \ldots, x_n, \ldots) = \text{the only point in } \bigcap_{n \geq 1} \varphi_{x_0,x_1} \varphi_{x_1,x_2} \cdots \varphi_{x_{n-1},x_n}(E_{x_n+1}).$$

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It is clear that \( \pi \) is a Hölder continuous surjective map. We will denote the composition \( \varphi_{x_0 x_1} \cdots \varphi_{x_{n-1} x_n} \) by \( \varphi_{x_0 \cdots x_n} \). Also, we assume all \( \varphi_{ij} \) to be \( C^1 \) diffeomorphisms and denote the derivative of \( \varphi_{ij} \) at a point \( x \) by \( T_x \varphi_{ij} \) or \( T \varphi_{ij}(x) \).

**Definition 1.1.** The \( j \)th Lyapunov number of a linear map \( L \), denoted by \( \alpha_j(L) \), is the square root of the \( j \)th largest eigenvalue of \( LL^* \), where \( L^* \) is the conjugate of \( L \). Write \( \omega(L) = \alpha_1(L) \cdots \alpha_{[\lambda]}(L) \alpha_{[\lambda]+1}(L)^{-1} \). For a set of construction diffeomorphisms \( \{ \varphi_{ij} \} \), define \( \lambda_t: \Sigma \to \mathbb{R} \) for each \( t \in [0, 1] \) and \( x = (x_0 x_1 \cdots) \in \Sigma \) by

\[
\lambda_t(x) = \log \alpha_1(T \varphi_{x_0 x_1}(\pi x_1)) + \cdots + \log \alpha_{[\lambda]}(T \varphi_{x_0 x_1}(\pi x_1)) \]

\[
+ (t - [t]) \log \alpha_{[\lambda]+1}(T \varphi_{x_0 x_1}(\pi x_1))
\]

\[
= \log \omega_t(T \varphi_{x_0 x_1}(\pi x_1)).
\]

Here "\( \log \)" is the natural logarithm.

The constructions and dimensions of self-similar sets have been studied by several authors under various restrictions. In this paper we relax the restrictions on construction diffeomorphisms to a \( \kappa \)-pinching condition, which is defined as follows.

**Definition 1.2.** We say that a \( C^1 \) homeomorphism \( \varphi_{ij} \) satisfies the \( \kappa \)-pinching condition if for all \( x \in E \) the derivatives satisfy \( \|T_x \varphi_{ij}\|^{1+\kappa} \cdot \|T_{\varphi_{ij}(x)}\varphi_{ij}^{-1}\| < 1 \).

**Remark.** If \( T_x \varphi_{ij} T_x \varphi_{ij}^* \) has eigenvalues \( \alpha_{1,ij}(x)^2 \geq \cdots \geq \alpha_{\lambda,ij}(x)^2 \) where \( T_x \varphi_{ij}^* \) denotes the conjugate of \( T_x \varphi_{ij} \), the numbers \( \alpha_{1,ij}(x), \ldots, \alpha_{\lambda,ij}(x) \) are Lyapunov numbers with \( 1 > \alpha_{1,ij}(x) \geq \cdots \geq \alpha_{\lambda,ij}(x) > 0 \). The pinching condition is equivalent to \( \alpha_{1,ij}(x)^{1+\kappa} < \alpha_{\lambda,ij}(x) \).

For the definition and properties of Hausdorff dimension, refer to [K]. Also, we use the definitions and notions of [W] in the discussion concerning topological pressure.

**Theorem 1.** Let \( \{ \varphi_{ij} \} \) be the \( C^1 \) construction diffeomorphisms for the self-similar set \( E \), satisfying the \( \kappa \)-pinching condition for some positive number \( \kappa \leq 1 \). Suppose the derivatives of all \( \{ \varphi_{ij} \} \) are Hölder continuous of order \( \kappa \). If \( t \) is the unique positive number such that the topological pressure \( P(\sigma, \lambda_t) = 0 \), then the Hausdorff dimension \( \text{HD}(E) \leq t \).

Let us recall the disjoint open set condition on the construction of self-similar sets; see [H]. It states that for each integer \( i \) from 1 to \( k \) there is a nonempty open set \( U_i \) such that

\[
\bigcup_{a_{ij}=1} \varphi_{ij}(U_j) \subset U_i \quad \text{and} \quad \varphi_{ij}(U_j) \cap \varphi_{ik}(U_k) = \emptyset \quad \text{if} \ j \neq k.
\]

For \( n \geq 0 \), denote \( U_n(x) = \varphi_{x_{0} x_{1}} \cdots \varphi_{x_{n-1} x_{n}}(U_{x_{n}}) \). It follows that \( E_i \subset \overline{U}_i \) and that the collection \( \{ U_n(x): x \in \Sigma \} \) is pairwisely disjoint for each fixed \( n \).

**Theorem 2.** Let \( \{ \varphi_{ij} \} \) be the \( C^1 \) construction diffeomorphisms for the self-similar set \( E \), satisfying both the \( \kappa \)-pinching condition for some positive number \( \kappa \leq 1 \), and the disjoint open set condition. Suppose the derivatives of all \( \{ \varphi_{ij} \} \) are Hölder continuous of order \( \kappa \). If \( t \) is the unique positive number
such that the topological pressure $P(\sigma, \lambda_1) = 0$, then the Hausdorff dimension $\text{HD}(E) \geq t/(1 + \kappa) - \kappa$.

**Remark.** We call a $C^1$ diffeomorphism $C^{1+\kappa}$ if its derivative is Hölder continuous of order $\kappa$. If we fix the construction to be $C^{1+\beta}$ for some $\beta > 0$ but let $\kappa \to 0$ for the $\kappa$-pinching condition, then our upper and lower bounds will coincide with the estimate for conformal cases in [B1].

Theorem 1 is proved in §2, and Theorem 2 is proved in §3. As a corollary of Theorems 1 and 2, in §4 we will also discuss some continuity in the $C^1$ topology of the Hausdorff dimension at conformal $C^{1+\kappa}$ constructions under disjoint open set condition. For discussions of the constructions of self-similar sets using similitudes and their dimensions, see [H] and [MW]. For the constructions using "conformal" contraction maps, see [B1]. Other related works can be found in [D, B2, F]. A similar result for basic sets in two dimensions can be found in [MM].

2. The upper bound

**Lemma 2.1.** If all construction diffeomorphisms $\varphi_{ij}$ are $C^{1+\kappa}$ and satisfy the $\kappa$-pinching condition, then for any $\epsilon > 0$, there exists $\delta > 0$ depending only on $\epsilon$, such that for all $x \in E$, all $a$ with $0 < a < \delta$, and all $x = (x_0, x_1, \ldots)$ in $\Sigma$, all integers $n > 0$, we have

$$
\varphi_{x_0\ldots x_n}(B(x,a)) \subset \varphi_{x_0\ldots x_n}(x) + (1 + \epsilon)^n T_x \varphi_{x_0\ldots x_n}(0,a).
$$

Here $B(x,a)$ denotes a ball of radius $a$ centering at $x$ in $\mathbb{R}^l$.

**Proof.** Using Taylor’s formula, for any $y, w \in \mathbb{R}^l$,

$$
\varphi_{x_0x_1}(y + w) = \varphi_{x_0x_1}(y) + T_y \varphi_{x_0x_1}(w) + r_{x_0x_1}(w, y).
$$

Since $E$ compact, we can find some constant $C > 0$ and $c > 0$, such that for all $y \in E$ and $w \in \mathbb{R}^l$ with $\|w\| \leq c$, we have $\|r_{x_0x_1}(w, y)\| < C\|w\|^{1+\kappa}$. We will set also $b = \min_{x \in E, i, j} \{\alpha_{i,j}(x)\}$, where $\alpha_{i,j}$ is the square root of the least eigenvalue of $T_x \varphi_{ij} T_x \varphi_{ij}$.

Fix any small $\epsilon > 0$. Since $E$ is compact and all construction diffeomorphisms satisfy the $\kappa$-pinching condition, without loss of generality we can assume $\epsilon$ to be so small that for all pairs $(i, j)$,

$$
\epsilon \alpha_{1,i,j}(x) < 1 \quad \text{for all } x \in E,
$$

$$
(1 + \epsilon)^{\kappa \alpha_{1,i,j}(x)} < 1 \quad \text{for all } x \in E.
$$

Pick $\delta > 0$, with $\delta < \min\{c, (b\epsilon/C)^{1/\kappa}\}$. Thus $\delta^\kappa < \epsilon \alpha_{1,i,j}(x)/C$ for all $x \in E$ and all pairs of $(i, j)$. Let $a \leq \delta$, and pick any $w \in \mathbb{R}^l$ with $\|w\| < a$.

For any $x$ in $E$, $\|r_{x_0x_1}(x, w, y)\| < C\|w\|^{1+\kappa} < Ca^{1+\kappa} \leq aC\delta^\kappa \leq \epsilon \alpha_{1,x_0x_1}(x)$ and thus $r_{x_0x_1}(x, w, y) \in \epsilon \alpha_{1,x_0x_1}(x)B(0, a)$. Since

$$
\epsilon \alpha_{1,x_0x_1}(x)B(0, a) \subset \epsilon T_x \varphi_{x_0x_1}B(0, a),
$$

it follows from (2.2) that

$$
\varphi_{x_0x_1}(x + w) = \varphi_{x_0x_1}(x) + T_x \varphi_{x_0x_1}(w) + r_{x_0x_1}(w, x)
$$

$$
\in \varphi_{x_0x_1}(x) + T_x \varphi_{x_0x_1}B(0, a) + \epsilon T_x \varphi_{x_0x_1}B(0, a)
$$

$$
= \varphi_{x_0x_1}(x) + (1 + \epsilon)T_x \varphi_{x_0x_1}B(0, a).
$$
This gives (2.1) for \( n = 1 \). Now the induction hypothesis gives

\[
\varphi_{x_0 x_1 \ldots x_n} B(x, a) = \varphi_{x_0 x_1 \varphi_{x_1 \ldots x_n} B(x, a)} 
\]

\[
\subset \varphi_{x_0 x_1} [\varphi_{x_1 \ldots x_n} (x) + (1 + \varepsilon)^{n-1} T_x \varphi_{x_1 \ldots x_n} B(0, a)].
\]

Using (2.2),

\[
\varphi_{x_0 x_1} [\varphi_{x_1 \ldots x_n} (x) + (1 + \varepsilon)^{n-1} T_x \varphi_{x_1 \ldots x_n} (w)] 
\]

\[
= \varphi_{x_0 \ldots x_n} (x) + (1 + \varepsilon)^{n-1} T \varphi_{x_0 \ldots x_n} (w) 
\]

\[
+ r_{x_0 x_1} ((1 + \varepsilon)^{n-1} T \varphi_{x_1 \ldots x_n} (w), \varphi_{x_1 \ldots x_n} (x)).
\]

Because of (2.3), \(|(1 + \varepsilon)^{n-1} T \varphi_{x_1 \ldots x_n} (w)| < \|w\| < a\), where \( w \in B(0, a) \).

Using (2.4), we have

\[
\|r_{x_0 x_1} ((1 + \varepsilon)^{n-1} T \varphi_{x_1 \ldots x_n} (w), \varphi_{x_1 \ldots x_n} (x))\| 
\]

\[
< C \|(1 + \varepsilon)^{n-1} T \varphi_{x_1 \ldots x_n} (w)\|^{1+\kappa} 
\]

\[
\leq C(1 + \varepsilon)^{n-1} \|T \varphi_{x_1 \ldots x_n} (w)\|^{1+\kappa} 
\]

\[
< (1 + \varepsilon)^{-1} a^\kappa C \|T \varphi_{x_1 \ldots x_n} (w)\|^{1+\kappa} 
\]

\[
< (1 + \varepsilon)^{-1} a^\kappa C \|T \varphi_{x_1 \ldots x_n} (w)\|^{1+\kappa} 
\]

\[
< \varepsilon (1 + \varepsilon)^{-1} a^\kappa C \|T \varphi_{x_1 \ldots x_n} (w)\|^{1+\kappa} 
\]

On the other hand \( T_x \varphi_{ij} B(0, a) \supseteq \alpha_{i,j} (x) B(0, a) \), and it follows that

\[
T_x \varphi_{x_0 \ldots x_n} B(0, a) \supseteq \alpha_{i,j} (x) B(0, a)
\]

Hence

\[
r_{x_0 x_1} ((1 + \varepsilon)^{n-1} T \varphi_{x_1 \ldots x_n} (w), \varphi_{x_1 \ldots x_n} (x)) \in \varepsilon (1 + \varepsilon)^{-1} T_x \varphi_{x_0 \ldots x_n} B(0, a).
\]

Therefore,

\[
\varphi_{x_0 x_1} [\varphi_{x_1 \ldots x_n} (x) + (1 + \varepsilon)^{n-1} T \varphi_{x_1 \ldots x_n} (w)] 
\]

\[
\in \varphi_{x_0 \ldots x_n} (x) + (1 + \varepsilon)^{n-1} T \varphi_{x_0 \ldots x_n} B(0, a) + \varepsilon (1 + \varepsilon)^{-1} T_x \varphi_{x_0 \ldots x_n} B(0, a)
\]

\[
\subset \varphi_{x_0 \ldots x_n} (x) + (1 + \varepsilon)^n T_x \varphi_{x_0 \ldots x_n} B(0, a).
\]

Thus (2.1) is true for \( n \). This completes the induction process. \( \Box \)

I have learned that Jiang [J] has a distortion lemma for a regular nonconformal semigroup, which is a semigroup of pinched contracting diffeomorphisms. His version is stronger than our version here. However, for our purpose of estimating Hausdorff dimensions, our version is strong enough.

**Proposition 2.2.** If all construction diffeomorphisms \( \varphi_{ij} \) are \( C^{1+\kappa} \) and satisfy the \( \kappa \)-pinching condition where \( 0 < \kappa \leq 1 \), and if the topological pressure \( P(\sigma, \lambda) < 0 \) where \( \sigma \) is the shift map in \( \Sigma \), then the Hausdorff dimension \( \text{HD}(E) \leq t \).

**Proof.** Choose small \( \varepsilon > 0 \) with \( P(\sigma, \lambda) < -2t \varepsilon \), satisfying both (2.3) and (2.4). By Lemma 2.1, there exists \( \delta > 0 \) such that (2.1) holds for all integer \( n > 0 \) and each \( x \in E \), when \( 0 < a < \delta \).

We fix \( a < \delta \) small enough, and a positive integer \( n \) big enough, such that (see [W] for notation) \( \log P_n (\sigma, \lambda, a) < -2nt \varepsilon \). Recall that \( \pi \) is Hölder continuous. Suppose that \( \gamma \) is the exponent such that there exists a constant \( D \)
with $|\pi(x) - \pi(y)| < D \cdot d(x, y)^\gamma$ for all $x$, $y$ in $\Sigma$. Fix $a' < \min\{D^{-1/\gamma}a^{1/\gamma}, a\}$. Pick $m$ with $2^{-m-1} < a' < 2^{-m}$. Let

$$K' = \{(x_0, \ldots, x_{m+n}) | \text{there exists } x \in \Sigma \text{ with } x = (x_0, \ldots, x_{m+n}, \ldots)\}.$$  

Choose for each word $(x_0, \ldots, x_{m+n})$ in $K'$ a point $x$ in $\Sigma$ with the initial of $x_0, \ldots, x_{m+n}$ to form a subset $K$ of $\Sigma$. The subset $K$ is $(n, a')$ separated and is maximal in the sense that one cannot add another point to $K$ such that it is still $(n, a')$ separated. Thus, the collection $\{\sigma^{-n}B(\sigma^n x, a') | x \in K\}$ is an open cover for $\Sigma$. Notice that $\pi x = \phi_{x_0, x_1} \pi \sigma x$. Since $\pi B(x, a') \subset B(\pi(x), a)$ and $\pi\{\sigma^{-n}B(\sigma^n x, a') | x \in K\} \subset \{\phi_{x_0, x_1, \ldots, x_n} B(\pi\sigma^n x, a) | x = (x_0, x_1, \ldots, x_n, \ldots) \in K\}$ follows, $\{\phi_{x_0, x_1, \ldots, x_n} B(\pi\sigma^n x, a) | x = (x_0, x_1, \ldots, x_n, \ldots) \in K\}$ is an open cover for $E = \bigcup_{i=1}^j E_i$.

Using (2.1) of Lemma 2.1,

$$\phi_{x_0, \ldots, x_n} B(\pi\sigma^n x, a) \subset \phi_{x_0, \ldots, x_n} (\pi\sigma^n x) + (1 + \varepsilon)^n T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} B(0, a).$$

The right side of (2.5) is an ellipsoid with axes $\{a(1 + \varepsilon)^n \alpha_j(T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n}) | 1 \leq j \leq l\}$. Pick $j$ with $j - 1 \leq t < j$. Then that ellipsoid can be covered by

$$C \cdot \alpha_1 \left( T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} \right) \cdots \alpha_j \left( T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} \right) / \alpha_j \left( T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} \right) = C \cdot \omega_{j-1} \left( T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} \right) \alpha_j^{-j+1} \left( T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} \right)$$

balls of radius $a(1 + \varepsilon)^n \alpha_j(T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n})$, where the constant $C > 0$ depends only on the dimension of $\mathbb{R}^l$. Now we calculate the Hausdorff $t$-measure of $E$, using the smaller balls of radius $a(1 + \varepsilon)^n \alpha_j(T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n}) < a$ to cover the open set $\phi_{x_0, \ldots, x_n} B(\pi\sigma^n x, a)$. If $\{P_i : i \in I\}$ is an open cover for $E$ where $P_i$ is a ball of radius $r_i$, then we define $|I| = \max_{i \in I} r_i$ and $\mu(a, t) = \inf_{|I| < a} \sum_{i \in I} r_i^t$.

We have

$$\mu(a, t) \leq \sum_{x \in K} C \omega_{j-1} \left( T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} \right) \alpha_j^{-j+1} \left( T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} \right)$$

$$= (1 + \varepsilon)^n a'C \sum_{x \in K} \omega_t \left( T_{\pi\sigma^n x} \phi_{x_0, \ldots, x_n} \right)$$

$$\leq (1 + \varepsilon)^n C \sum_{x \in K} \omega_t \left( T_{\pi\sigma x} \phi_{x_0, x_1} \right) \omega_t \left( T_{\pi\sigma x} \phi_{x_1, x_2} \right) \cdots \omega_t \left( T_{\pi\sigma x} \phi_{x_{n-1}, x_n} \right)$$

$$= (1 + \varepsilon)^n C \sum_{x \in K} \exp[\lambda_t(x) + \lambda_t(\sigma x) + \cdots + \lambda_t(\sigma^{n-1} x)]$$

$$\leq (1 + \varepsilon)^n C P_n(\sigma, \lambda_t, a') \leq (1 + \varepsilon)^n C P_n(\sigma, \lambda_t, a)$$

$$\leq C \exp(nt\varepsilon) \exp(-2nt\varepsilon) \to 0,$$

as $n \to \infty$. Thus $\mu(a, t) = 0$. Since $a$ can be arbitrarily small, $\mu(t) = 0$. It follows that $\text{HD}(E) \leq t$.  

**Proof of Theorem 1.** $P(\sigma, \lambda_t)$ is a decreasing function of $t$'s since $E$ is compact and $\lambda_t$ is strictly decreasing with respect to $t$. So there is only one real number $t$ such that $P(\sigma, \lambda_t) = 0$. Also, the unique $t$ with $P(\sigma, \lambda_t) = 0$ is equal to $\inf\{t : P(\sigma, \lambda_t) < 0\}$. Consequently, we have $\text{HD}(E) \leq t$ where $P(\sigma, \lambda_t) = 0$.  

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3. The Lower Bound

Proof of Theorem 2. Notice that for each $t$, the map $\lambda_t$ is H"older continuous on $\Sigma$. So there exists an equilibrium state $\mu$ for $\lambda_t$, in the sense that

$$P(\sigma, \lambda_t) = h_\mu(\sigma) + \int \lambda_t d\mu.$$

Fix any $\rho > 0$, let us estimate the $\mu$-measure of a ball $B(z, \rho)$ centered at $z$ with radius $\rho$. For each $x \in \Sigma$ choose the unique $n = n(x) \geq 0$ such that the diameters satisfy

$$\text{diam}(U_n(x)) \leq \rho < \text{diam}(U_{n-1}(x)).$$

Lemma 3.1. There exists a constant $c > 0$ such that for all $x \in \Sigma$, the open set $U_n(x)$ is contained in a ball of radius $\rho$ and contains a ball of radius $c\rho^{1+\kappa}$.

Proof. It is clear that $U_n(x)$ is contained in a ball of radius $\rho$. Since the radius of $U_n(x)$ decreases to 0 as $n$ grows to infinity, without loss of generality we can assume the maximum diameter $R$ of all $U_i$ is less than the number $\delta$ given in Lemma 2.1. Also pick $r$ small enough that each $U_i$ contains a ball of radius $r$. Then $U_n(x)$ contains a ball of radius

$$r \cdot \alpha_{1,x_0,x_1}(\pi \sigma x) \cdots \alpha_{1,x_{n-1},x_n}(\pi \sigma^n x) > r \cdot \alpha_{1,x_0,x_1}(\pi \sigma x) \cdots \alpha_{1,x_{n-1},x_n}(\pi \sigma^n x).$$

But on the other hand

$$\rho \leq \text{diam}(U_{n-1}(x)) \leq \alpha_{1,x_0,x_1}(\pi \sigma x) \cdots \alpha_{1,x_{n-1},x_n}(\pi \sigma^{n-1} x) R,$$

which implies that $\alpha_{1,x_0,x_1}(\pi \sigma x) \cdots \alpha_{1,x_{n-1},x_n}(\pi \sigma^n x) \geq \alpha_1 \rho / R$ where the constant $\alpha_1 = \min_{y \in E, i,j} \{\alpha_{1,i,j}(y)\} > 0$ does not depend on either $n$ or $x$. Therefore $U_n(x)$ contains a ball of radius $> r\rho^{1+\kappa} \alpha_1^{1+\kappa} / R^{1+\kappa}$. Writing $c = \alpha_1^{1+\kappa} / R^{1+\kappa}$ a constant, $U_n(x)$ contains a ball of radius $c\rho^{1+\kappa}$ as desired. □

For two points $x, y \in \Sigma$, since the construction maps satisfy the open set condition, $U_n(x)$ and $U_n(y)$ are either equal or disjoint. Let $\Gamma \subset \Sigma$ be a subset such that $\{U_n(x) \mid x \in \Gamma\}$ is a disjoint collection which contains all $U_n(x)$ for $x \in \Sigma$. Notice that $\{\overline{U}_n(x) \mid x \in \Gamma\}$ covers $E$.

Lemma 3.2 (similar to [H, 5.3(a)]). At most $3^l c^{-l} \rho^{-k l}$ of $\{\overline{U}_n(x) \mid x \in \Gamma\}$ can meet $B(z, \rho)$.

Proof. Suppose that $\overline{V}_1, \ldots, \overline{V}_m$ in $\{\overline{U}_n(x) \mid x \in \Gamma\}$ meet $B(z, \rho)$. Then each of them is a subset of $B(z, 3\rho)$. By the definition of $\Gamma$ the sets in the collection $\{\overline{U}_n(x) \mid x \in \Gamma\}$ are disjoint. Comparing the volumes we have

$$m J c^{-l} \rho^{l(1+\kappa)} \leq J m c^{-l} \rho^{l} \rho^{l(1+\kappa)}$$

where $J$ is the volume of a unit ball in $R^l$. Hence

$$m \leq 3^l c^{-l} \rho^{-k l}.$$ □

Let $C_n(x) = \{y = (y_0, y_1, \ldots) \in \Sigma \mid y_0 = x_0, \ldots, y_n = x_n\}$ be the $n$ cylinder. Recall that $\mu$ is a Gibbs measure (see [Bo] for a discussion or [B2] for a summary). There exists a constant $d > 0$, with

$$\mu(C_n(x)) \in [d^{-1}, d] \cdot \exp(-P(\sigma, \lambda_t) n + S_n \lambda_t(x)),$$

for each cylinder $C_n(x)$ in $\Sigma$. 

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Thus \( \mu(C_n(x)) \in [d^{-1}, d] \cdot \exp(S_n \lambda_t(x)) \), since \( P(\sigma, \lambda_t) = 0 \). So,

\[
\mu(C_n(x)) \leq d \exp S_n \lambda_t(x)
\]

\[
\leq d[\alpha_1, x_0x_1(\pi\sigma x) \cdots \alpha_1, x_{-1}x_n(\pi\sigma^n x)]
\]

\[
\leq d[\alpha_i, x_0x_1(\pi\sigma x) \cdots \alpha_i, x_{-1}x_n(\pi\sigma^n x)]^{(1+\kappa)}
\]

\[
\leq d \cdot \text{diam}(U_{n(x)})^r(1+\kappa).
\]

Hence if \( n = n(x) \) we obtain

\[
\mu(C_n(x)) \leq \frac{d^{r/(1+\kappa)}}{r^{r/(1+\kappa)}}.
\]

Noticing \( \pi C_n(x) \supset U_n(x) \cap E \), by Lemma 3.2,

\[
\pi_*\mu(B(z, r)) \leq [3e^{-l(r-1)/(1+\kappa)}]^{p/(1+\kappa)} r^l(1+\kappa) - l\kappa.
\]

By the Frostman lemma (see [K] for a proof), \( \text{HD}(E) \geq t/(1 + \kappa) - l\kappa \).

4. Some continuity of the Hausdorff dimension in \( C^1 \) topology

The construction of the self-similar set \( E_\varphi \) depends on the contracting diffeomorphisms \( \{\varphi_{ij}\} \). Now let us fix \( 0 < \beta \leq 1 \), and consider a \( C^1 \) perturbation to a \( C^{1+\beta} \) conformal construction with diffeomorphisms \( \{\varphi_{ij}\} \), and obtain another matrix of contracting diffeomorphisms \( \{\psi_{ij}\} \), which is not necessarily conformal. Denote the new self-similar set for \( \psi \) by \( E_\psi \). Define 

\[
d_{C^1}(\varphi, \psi) = \max_{i,j} d_{C^1}(\{\varphi_{ij}, \psi_{ij}\}),
\]

where the latter \( d_{C^1} \) is the \( C^1 \) metric. Note that for any \( \kappa < \beta \), when \( \psi \) is sufficiently \( C^1 \) close to \( \varphi \), \( \psi \) must be \( C^{1+\kappa} \) and also \( \kappa \)-pinched. The following theorem is a corollary of Theorems 1 and 2, which states that at a \( C^{1+\beta} \) conformal construction satisfying the open set condition for self-similar sets, the Hausdorff dimension \( \text{HD}(E_\psi) \) depends continuously on \( \{\psi_{ij}\} \) in \( C^1 \) topology.

**Theorem 4.1.** Let \( \{\varphi_{ij}\} \) be a matrix of \( C^{1+\beta} \) conformal construction diffeomorphisms for the self-similar set \( E_\varphi \), satisfying the open set condition. For any \( \epsilon > 0 \), there exists \( \delta > 0 \), such that for any \( C^{1+\beta} \) construction \( \psi \) satisfying the open set condition, with \( d_{C^1}(\varphi, \psi) < \delta \), we have \( |\text{HD}(E_\varphi) - \text{HD}(E_\psi)| < \epsilon \).

**Proof.** Let \( \lambda_{\varphi,s}(x) = \log \omega_s(T\varphi_{x_0x_1}(\pi\sigma x)) \) and \( \lambda_{\psi,s}(x) = \log \omega_s(T\psi_{x_0x_1}(\pi\sigma x)) \) be two real functions on \( \Sigma \) as defined in Definition 1.1. Let \( t \) be such that \( P(\sigma, \lambda_{\varphi,t}) = 0 \). Because \( \varphi_{ij} \)'s are conformal, the Hausdorff dimension of \( E_\varphi \) equals \( t \). Also, remark that \( P(\sigma, \lambda_{\psi,t+\epsilon}) < 0 \) for any \( \epsilon > 0 \).

Now fix any \( \epsilon > 0 \). Let \( \kappa = \min(\beta, \epsilon/4l) \) and let

\[
e' = \frac{1}{2} \min\{-P(\sigma, \lambda_{\varphi,t+\epsilon}), P(\sigma, \lambda_{\varphi,t-\epsilon/4})\} > 0.
\]

Since \( \varphi \) is \( C^{1+\beta} \) and conformal, there is \( \delta > 0 \) such that a \( C^{1+\beta} \) diffeomorphism \( \psi \) is \( C^{1+\kappa} \) and \( \kappa \)-pinched with \( |\lambda_{\varphi,s}(x) - \lambda_{\psi,s}(x)| < \epsilon' \) for all \( s \in [0, l] \), if \( d_{C^1}(\varphi, \psi) < \delta \).

Then \( P(\sigma, \lambda_{\psi,t+\epsilon}) < P(\sigma, \lambda_{\varphi,t+\epsilon} + \epsilon') \leq P(\sigma, \lambda_{\varphi,t+\epsilon} + \epsilon') < 0 \). So \( \text{HD}(E_\psi) \leq t + \epsilon = \text{HD}(E_\varphi) + \epsilon \), by Proposition 2.2.

On the other hand, by (4.1), when \( d_{C^1}(\varphi, \psi) < \delta \), we have

\[
P(\sigma, \lambda_{\psi,t-\epsilon/4}) > P(\sigma, \lambda_{\varphi,t-\epsilon/4} - \epsilon') \geq P(\sigma, \lambda_{\varphi,t-\epsilon/4} - \epsilon') > 0.
\]

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So we have some \( s > t - \varepsilon/4 \) with \( P(\sigma, \lambda_{\psi}, s) = 0 \) since \( P(\sigma, \lambda_{\psi}, s) \) is strictly decreasing with respect to \( s \). By Theorem 2, \( HD(E_\psi) \geq s/(1 + \kappa) - \kappa \geq s/(1 + \varepsilon/4) - l/(\varepsilon/4) > s(1 - \varepsilon/4) - \varepsilon/4 > s - \varepsilon/2 > t - \varepsilon \). It then follows that \( HD(E_\psi) > t - \varepsilon = HD(E_\varphi) - \varepsilon \). \( \square \)

We say a construction \( \varphi \) with diffeomorphisms \( \{\varphi_{ij}\} \) satisfies the strong open set condition if there are open sets \( U_1, \ldots, U_l \) in \( \mathbb{R}^l \) with \( \varphi_{ij}(U_j) \subset U_i \) for all \( i, j \). If the construction \( \varphi \) satisfies the strong open set condition, then \( \psi \) must also satisfy the strong open set condition if it is \( C^1 \) close enough to \( \varphi \). Thus we have obtained an immediate corollary of the above Theorem 4.1:

**Corollary 4.2.** Let \( \{\varphi_{ij}\} \) be a matrix of \( C^{1+\beta} \) conformal construction diffeomorphisms for the self-similar set \( E_\varphi \), satisfying the strong open set condition. For any \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that for any \( C^{1+\beta} \) construction \( \psi \) with \( d_{C^1}(\varphi, \psi) < \delta \), we have \( |HD(E_\varphi) - HD(E_\psi)| < \varepsilon \).

Finally we have a remark on the continuity of the Hausdorff dimension in \( C^1 \) topology at nonconformal constructions.

**Remark 4.3.** The following example shows that if the "conformal" condition for the construction diffeomorphisms \( \{\varphi_{ij}\} \) fails, then the results in Theorem 4.1 and Corollary 4.2 can be false. The example is derived from Example 9.10 of Falconer [F, pp. 127-128].

Let \( S, T_\lambda : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by

\[
S(x, y) = (x/2, y/3 + 2/3), \quad T_\lambda(x, y) = (x/2 + \lambda, y/3)
\]

where \( \lambda \in [0, 1/2) \) and \( (x, y) \in \mathbb{R}^2 \). Let \( \varphi_{11} = \varphi_{21} = S, \varphi_{12} = \varphi_{22} = T_0 \). Take \( \psi_\lambda = \{\psi_{ij, \lambda}\} \) where \( \psi_{11, \lambda} = \psi_{21, \lambda} = S \) and \( \psi_{12, \lambda} = \psi_{22, \lambda} = T_\lambda \). The strong open set condition is met for \( \{\varphi_{ij}\} \). In fact, if we let \( U_1 = U_2 = (-1/8, 9/8)^2 \subset \mathbb{R}^2 \) then \( \varphi_{ij}(U_j) \subset U_i \).

Let \( E_\varphi, E_\psi \) be the self-similar sets for \( \varphi \) and \( \psi_\lambda \). Considering the projection of \( E_{\psi_\lambda} \) to the \( x \)-axis, one knows that \( HD(E_{\psi_\lambda}) \geq 1 \) for \( \lambda > 0 \). But \( E_\varphi \) is a Cantor set contained in the \( y \)-axis with the Hausdorff dimension \( HD(E_\varphi) = (\log 2)/\log 3 < 1 \). Since \( d_{C^1}(\psi_\lambda, \varphi) = \lambda \), letting \( \lambda \to 0 \) we know the Hausdorff dimension is not continuous at \( \varphi \). We notice that \( \{\varphi_{ij}\} \) and \( \{\psi_{ij, \lambda}\} \) are all \( 3/4 \) pinched.

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**References**


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