THE HAUSDORFF DIMENSION OF THE NONDIFFERENTIABILITY SET OF THE CANTOR FUNCTION IS $[\ln(2)/\ln(3)]^2$

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Abstract. The main purpose of this note is to verify that the Hausdorff dimension of the set of points $N^*$ at which the Cantor function is not differentiable is $[\ln(2)/\ln(3)]^2$. It is also shown that the image of $N^*$ under the Cantor function has Hausdorff dimension $\ln(2)/\ln(3)$. Similar results follow for a standard class of Cantor sets of positive measure and their corresponding Cantor functions.

The Hausdorff dimension of the set of points $N^*$ at which the Cantor function is not differentiable is $[\ln(2)/\ln(3)]^2$.

Chapter 1 in [5] provides a nice introduction to Hausdorff measure and dimension; references [5–7] pursue the topic. We begin our proof with some notation and discussion. Let $C$ denote the Cantor set. Let $N^+$ ($N^-$) denote the set of points at which the Cantor function does not have a right side (left side) derivative, finite or infinite. Then $N^* = N^+ \cup N^- \cup \{t: t$ is an end point of $C\}$ denotes the nondifferentiability set of the Cantor function. Although we will assume familiarity with [4], where Eidswick characterized $N^*$, some material is repeated for completeness.

A number $t$ in $C$ has a ternary representation $t = (t_1, \ldots, t_i, \ldots)$, where $t_i = 0$ or 2.

Let $z(n)$ denote the position of the $n$th zero in the ternary representation of $t$;

(1a) If $t \in N^+$, then $\limsup \{z(n + 1)/z(n)\} \geq \ln(3)/\ln(2)$;

(1b) If $\limsup \{z(n + 1)/z(n)\} > \ln(3)/\ln(2)$, then $t \in N^+$.

Let $m_d$ denote the $d$-dimensional Hausdorff measure, and put $r = \ln(2)/\ln(3)$.

We will compute the Hausdorff dimension of $N^*$ by verifying

(A) If $1 > d > r^2$, then $m_d N^* = 0$.

(B) If $d < r^2$, then $m_d N^* \geq K_d > 0$; $K_d$ will be specified later for a sequence of $d$'s increasing to $r^2$.

Condition (A) will be verified for each $d$ satisfying the inequalities $1 > d > r^2$ by constructing a set $E$ (depending on $d$) which contains $N^*$ and satisfies the equation $m_d E = 0$. To verify (B), we will consider a sequence $\{d_n\}$ of
$d$'s increasing to $r^2$; for each $d$ in the sequence, we will construct a subset $E^*$ of $N^*$ with $m_d(E^*) > 0$, which implies $m_h(N^*) > m_h(E^*) = \infty$ for $h < d$.

(A) implies $m_h(N^*) = 0$ for $h > r^2$, and (B) implies $m_h(N^*) = \infty$ for $h < r^2$.

Consequently, the Hausdorff dimension of $N^*$ is equal to $r^2$.

**Verification of (A).** We will use sets $E_k = \{t: t_k = 0$ and $t_i = 2$ for $k < i \leq u_k\}$, where $u_k$ will be specified below.

Fix $d > r^2$. We will define a positive integer $n$ (depending on $d$) and $u_k$ for $k \geq n$ so

1. \(N^+ \subseteq \bigcup_{k \geq m} E_k, m \geq n: N^+ \subseteq \limsup \{E_k\} = E^*\)

and

2. \[
\frac{2^k}{(3 u_k)^d} \leq k^{-2}: k \ln(2) - du_k \ln(3) \leq -2 \ln(k): \\
r + (2/\ln(3))(\ln(k)/k) \leq d(u_k/k).
\]

The required strings of 2’s in the points of $E^*$ will be short enough to apply (1a) to verify (2), and they will be long enough to satisfy (3). Since $d > r^2$, put $d = r(r + t)$, where $t > 0$. Then $t = (d - r^2)/r < 1/a^2$. Choose $n \geq 3$ so that

3. \[
\ln(n)/n < t/4.
\]

Then $\ln(m)/m$ is decreasing for $m \geq n$ and $1/n < t/4$. Thus, for $k \geq n$ we can choose $u_k$ so that

4. \[
r^{-1} - t/2 < u_k/k < r^{-1} < t/4.
\]

According to (4) and the first inequality in (5), for $k \geq n$,

\[
r + (2/\ln(3))(\ln(k)/k) < r + 2(\ln(k)/k) < r + t/2
\]

so (3) is satisfied for $k \geq n$.

Referring to (1a) and the second inequality in (5), (2) is satisfied. To show that $m_d(\limsup \{E_k\}) = m_d(E^*) = 0$, it suffices to observe that since each $E_i$ can be covered with $2^{i-1}$ intervals of length $3^{-u_i}$, then

\[
m_d(E^*) \leq \lim_k \sum_{i \geq k} 2^i/(3^{u_i})^d \leq \lim_k \sum_{i \geq k} i^{-2} = 0.
\]

Consequently, $m_d N^+ = 0$. Similarly, $m_d N^- = 0$; thus, $m_d N^* = 0$.

**Verification of (B).** For $d = rsv$, where $s = (n - 1)/n$ and $v = rs$, we will construct a subset $E \subseteq N^+$ with $m_d E \geq K = PQR$, where $P$ and $Q$ are positive numbers that will be defined later and $R$ is a positive constant relating $m_d$ and the equivalent $d$-dimensional net measure $(\text{ter})_d$ obtained by requiring covers to be composed of ternary intervals $[a, b] = [i/3^k, (i + 1)/3^k]$ (which we call $k$-intervals) according to the inequality $m_d \geq R(\text{ter})_d$. The existence of $R$ follows from a variation on a theme of Besicovitch that is discussed in [5, §5.1; 7, Chapter 2, §7.1]; we are using closed ternary intervals, but only countably many end points are involved in intersections. Since $m_d \geq R(\text{ter})_d$, we verify (B) by establishing the inequality $(\text{ter})_d E \geq PQ$ below.

Covers are required to be ternary covers in the following discussion.
We begin by describing a generic set $E$ of the type to be used; $E$ corresponds to a sequence $0 < k_1 < u_1 < k_2 < u_2 < \cdots$ of positive integers as follows:

$$E = \{t = (t(1), t(2), \ldots): t(k_i) = 0 \text{ and } t(k) = 2 \text{ for } k_i < k \leq u_i, \; i \geq 1\}.$$  

The set $E$ is a closed subset of $C$ and is composed of non-end-points of $C$.

When $k_i \leq k \leq u_i$, $k$ is a fixed choice (for $E$); otherwise, $k$ is a free choice.

The strings of fixed choices will be long enough to make the points in $E$ satisfy (1b), and the strings of free choices will be long enough to assure $P > 0$ and $Q > 0$.

Let $F(p, q)$ denote the number of free choices $k$ with $p < k \leq q$.

Because of [4, Theorem 1] and the fact that

$$\liminf(3^z(n)/2^{z(n+1)}) \leq \liminf(3^{k_i}/2^{u_i}),$$

$E$ is a subset of $N^+$ if $\inf_j(u_i/k_i) > \ln 3/\ln 2$. In particular, recalling the definition of $v$, $E$ is a subset of $N^+$ if $u_i = v^{-1}k_i + r_i$, where $0 \leq r_i < 1$.

Let $\{[a_j, b_j]\}$ be a ternary cover of $E$. Since $E$ contains no end point of $C$, $\{(a_j, b_j)\}$ is an open cover of $E$; $E$ is compact, so we restrict attention to a finite subcover. We can also require $b_j \leq a_{j+1}$ and that $[a_j, b_j] \cap E$ be nonempty. Let $3^{-w} = \min\{b_j - a_j\}$. For $k > w$, a $k$-interval $U = [i/3^k, (i + 1)/3^k]$ intersects at most one $(a_j, b_j)$; if this intersection is nonempty, then $U \subseteq [a_j, b_j]$.

The $k_i$'s and $u_i$'s considered below are all $> w$. To prove $(\text{ter})_dE \geq PQ$, it suffices to specify positive constants $P$ and $Q$ satisfying

(C) $m_q[a_j, b_j] \geq P$ (number of $u_i$-intervals in $[a_j, b_j]) 3^{-u_i}$

(D) (number of $u_i$-intervals which intersect $E) 3^{-u_i}d \geq Q$.

Letting $[i/3^k, (i + 1)/3^k]$ denote a generic $[a_j, b_j]$, we rewrite (C) and (D) as

(C) $3^{-kd} \geq P2^F(k, k_i)/3^{u_i}$

(D) $2^F(0, k_i)/3^{u_i}d \geq Q$.

Define $u_i$ by the equation $u_i = v^{-1}k_i + r_i$, $0 \leq r_i < 1$. Thus, the points in $E$ satisfy (1b).

Define $k_{i+1}$ by specifying $F(0, k_{i+1}) = s(k_{i+1} + s_{i+1})$, where $0 \leq s_{i+1} < 1$ and $s_{i+1}$ is minimal. Such a choice is possible because for $1 < f < k$,

$$f/k - (f - 1)/(k - 1) = (k - f)/[k(k - 1)] < k^{-1}.$$  

This definition of $k_{i+1}$ provides enough free choices to assure $P > 0$ and $Q > 0$.

Verification of (C).

$$3^{-kd} \geq P2^F(k, k_i)/3^{u_i}d \Leftrightarrow 3^{(u_i-k)d} \geq P2^F(k, k_i)$$

$$\Leftrightarrow 2^F(u_i-k)^{3u_i}P2^F(k, k_i) \Leftrightarrow 1 \geq P2^{F(k, k_i) - vs(u_i-k)}.$$  

Put $h(k, i) = F(k, k_i) - sv(u_i - k)$. If $k_j \leq k \leq u_j$, then $h(k, i) \leq h(u_j, i)$; and if $u_{j-1} < k < k_j$, then $h(k, i) < h(u_{j-1}, i)$. Thus, it suffices to show that $h(u_j, i)$ is bounded for $j < i$.

$$F(u_j, k_i) - sv(u_i - u_j)$$

$$= [(sk_i + s_i) - (sk_j + s_j)] - sv[(v^{-1}k_i + r_i) - (v^{-1}k_j + r_j)]$$

$$= [s_i - s_j] - sv[r_i - r_j] < 1 + sv.$$  

Hence, we put $P = 2^{-(1+sv)}$.  

Verification of (D). \( F(0, k_i) = su/k_i + s_i, \ u_i d = (v^{-1}k_i + r_i)rs = rs(k_i + vr_i), \) and \( 3' = 2; \) consequently, \( 2F(0, k_i)/3u_d = 2(s_i - sv/sr_i) \geq 2^{sv} . \) Thus, putting \( Q = 2^{-sv} , \) we have shown that the Hausdorff dimension of \( N^* \) is \( r^2 . \)

Now we can get some free results about Hausdorff dimension. Denote the Cantor function by \( \phi \).

**The Hausdorff Dimension of** \( \phi(N^*) \) **is** \( \ln(2)/\ln(3). \)

This result follows straightforwardly from our previous work because the binary representation of \( \phi(t) \) is obtained by replacing the 2’s in the ternary representation of \( t \) by 1’s. Consequently, since \( 3' = 2 \) and intervals of length \( 3^{-k} \) correspond to intervals of length \( 2^{-k} \) when we go from the ternary representation of \( t \) to the binary representation of \( \phi(t) \), we can replace \( r^2 \) by \( r \) and \( (\text{ter})_d \) by \( (\text{bin})_d \) and modify the preceding arguments appropriately to verify that the Hausdorff Dimension of \( \phi(N^*) \) is \( \ln(2)/\ln(3). \)

Referring to [1–3], denote the standard Cantor set of measure \( 1 - \lambda \) by \( C_\lambda, \ 0 < \lambda < 1; \) denote the corresponding Cantor functions by \( \phi_\lambda \) and the corresponding nondifferentiability sets by \( N^*_\lambda. \)

The sets \( N^*_\lambda \) and \( \phi_\lambda(N^*_\lambda) \) have Hausdorff dimension \( \ln(2)/\ln(3), \ 0 < \lambda < 1. \)

This assertion follows from the descriptions of the \( \phi_\lambda \)'s given in [1], Theorems 2 and 3 in [3], and the results previously established in this note. Intervals generated in the description of \( C_\lambda \) as an intersection of finite unions of \( 2^k \) intervals of length \( L_k \) have \( L_k = (1 - \lambda)2^{-k} + \lambda 3^{-k} . \) The binary part of \( L_k \) overwhelms the ternary component; thus, variations of the arguments used to compute the Hausdorff dimension of \( \phi(N^*) \) suffice here.

**References**


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