INJECTIVE MORPHISMS OF AFFINE VARIETIES

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Abstract. In this note an elementary proof that every injective morphism from an affine variety into itself is necessarily surjective is given.

1. Introduction

Let $K$ be any algebraically closed field and $V$ an algebraic variety defined over $K$. It is known that any injective morphism from $V$ into itself is necessarily surjective [7, Proposition (10.4.11), p. 103; 1; 2; 6]. Borel remarked that Shimura had a proof of this theorem by reduction modulo $p$ [6]. It seems that such a proof has not been published yet. The purpose of this note is to provide an elementary proof along this line when $V$ is an affine variety. What we shall prove is the following

Theorem. Let $K$ be any algebraically closed field, $V$ an affine variety defined over $K$, and $\phi: V \to V$ a morphism from $V$ into itself. If $\phi$ is injective, then it is surjective.

By [2, p. 3] the general case when $V$ is any algebraic variety follows from the above theorem. In fact, we shall present a proof for the general case in the appendix using Shimura's reduction theory, although we do not know whether this proof is what Shimura had in mind. For additional information when $V$ is an affine space see [10; 5; 3, (2.1) Theorem].

2. The proof of the theorem

The essence of our proof goes back to an idea of Shafarevich about $p$-group actions on affine spaces [4, Lemma; 8, Theorem 4.1].

Let $V$ be an affine variety in $\mathbb{A}^n$, the affine $n$-space. Denote the polynomial ring of $n$ variables over $K$ by $K[X_1, \ldots, X_n]$. Let $I$ be the defining ideal of $V$ and $g_1, g_2, \ldots, g_s$ a set of generators of $I$. Denote the coordinate ring of $V$ by

$$R := K[X_1, \ldots, X_n]/I = K[x_1, \ldots, x_n],$$
where $g(X)$ is regarded as an element in $K[X_1, \ldots, X_n]$, while $g(x)$ is regarded as an element in $R$. Points in $V$ or $A^n$ will be denoted by $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ or simply by $a$, $b$.

Let $\phi$ be a morphism from $V$ into $V$ given by

$$\phi: V \rightarrow V$$

$$a \mapsto (f_1(a), f_2(a), \ldots, f_n(a))$$

where $f_1(x), \ldots, f_n(x) \in R$. Let $V \times V$ be the product space of $V$ with itself, and consider the morphism

$$\Phi: V \times V \rightarrow A^n$$

$$(a, b) \mapsto (f_1(a) - f_1(b), f_2(a) - f_2(b), \ldots, f_n(a) - f_n(b))$$

Denote the diagonal of $V \times V$ by

$$\Delta := \{(a_1, \ldots, a_n, a_1, \ldots, a_n) \in V \times V : a_i \in K\}.$$

It is clear that

$\phi$ is injective

$$\Leftrightarrow \Phi^{-1}(0, \ldots, 0) = \Delta.$$

$$\Leftrightarrow$$

there exists an integer $m$ such that

$$(1) \quad (x_i - y_i)^m = \sum_{j=1}^{n} h_{ij}(x, y)\{f_j(x) - f_j(y)\} \quad \text{for } 1 \leq i \leq n$$

for some $h_{ij}(x, y) \in R \otimes_K R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where we identify $x_i$ and $y_j$ with $x_i \otimes 1$ and $1 \otimes y_j$ in $R \otimes_K R$.

Suppose that $\phi$ is not surjective and $c = (c_1, \ldots, c_n)$ is not in $\phi(V)$. Then the system of equations

$$f_1(x) = c_1, \ldots, f_n(x) = c_n$$

has no solution in $V$. Therefore, by Hilbert's Nullstellensatz, there exist $h_1(x), \ldots, h_n(x) \in R$ so that

$$(2) \quad \sum_{i=1}^{n} h_i(x)\{f_i(x) - c_i\} = 1.$$  

Collect all the coefficients of $g_1(X), \ldots, g_s(X), f_1(X), \ldots, f_n(X)$, $h_{ij}(X, Y)$, $h_i(X)$, and all the $c_1, c_2, \ldots, c_n$. Call this set $\{d_1, d_2, \ldots, d_t\}$. Define a subring $S$ of $K$ by

$$S := \begin{cases} 
Z[d_1, d_2, \ldots, d_t] & \text{if char } K = 0, \\
Z_p[d_1, d_2, \ldots, d_t] & \text{if char } K = p > 0. 
\end{cases}$$

By Nagata's version of Noether's normalization lemma, there is a maximal ideal $M$ of $S$ so that $k := S/M$ is a finite field [9].

Let $W$ be the affine $n$-space over $k$ and $W_0$ the affine variety in $W$ defined by

$$W_0 = \{a \in W : \overline{g}_1(a) = \cdots = \overline{g}_s(a) = 0\},$$

where $\overline{g}$ is the image of $g$ when passing from $S$ onto $k$. 

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Thus we obtain an injective morphism \( \phi_0 \) given by

\[
\phi_0: W_0 \to W_0
\]

\[
a \mapsto (f_1(a), \ldots, f_n(a)),
\]

which is not surjective because the image of \( c = (c_1, \ldots, c_n) \) is in \( W_0 \) and both formulae (1) and (2) still hold when passing to \( k \). Remember, both \( W \) and \( W_0 \) are finite sets. Thus we get a one-to-one but not onto map from a nonempty finite set to itself—a contradiction.

**Appendix**

In this appendix we shall establish the theorem when the variety is any quasi-projective variety by applying Shimura's theory of reduction modulo \( p \) of algebraic varieties [11; 12, Appendix; 13, §9].

We recall some fundamental facts of Shimura's reduction theory. Let \( K \) and \( K' \) be two fixed universal domains. We only deal with specializations defined on a subfield of \( K \) taking values in \( K' \). Suppose that \( k \) and \( k' \) are subfields of \( K \) and \( K' \), respectively, \( \lambda: k \to k' \) is a specialization from \( k \) onto \( k' \), and \( V \) is a quasi-projective algebraic variety defined over \( k \). Then \( V \) has a unique specialization over \( \lambda \), which we denote by \( \overline{V} \). The notion of \( \lambda \)-simple varieties is introduced in [11, p. 163; 13, p. 83]. For a \( \lambda \)-simple quasi-projective variety \( V \), the specialization of \( V \) over \( \lambda \) preserves inclusion, sum, intersection-product, direct product, and projection [11, Theorems 17, 18, 19; 13, Proposition 1]. Moreover, if \( V \) is irreducible and \( x \) is a generic point over \( k \), then, as a point set, \( \overline{V} \) is equal to the set of all specializations of \( x \) over \( \lambda \) (into the universal domain \( K' \)) [12, Lemma 3].

Now we may start to prove

**Theorem A.** Let \( K \) be a universal domain, \( V \) any quasi-projective variety over \( K \), and \( \varphi: V \to V \) a morphism from \( V \) into itself. If \( \varphi \) is injective, then it is surjective.

**Proof.** As in the proof of §2, define a subring \( S \) of \( K \) in a similar way so that both \( V \) and \( \varphi \) are defined over \( k_0 \) and \( V \) has a \( k_0 \)-rational point, where \( k_0 \) is the quotient field of \( S \). By [12, Lemma 6] adjoin a finite number of nonzero elements of \( k_0 \) and their inverses to \( S \) so as to assure \( \lambda \)-simplicity. By abuse of language we still denote by \( S \) this enlarged finitely generated ring.

Again by Noether's normalization lemma, find a homomorphism \( \lambda: S \to k_0' \), where \( k_0' \) is some finite field. Extend \( \lambda \) to a specialization of \( k_0 \); we still call it \( \lambda \). Let \( K' \) be a fixed universal domain containing \( k_0' \). Then \( \overline{V} \), the specialization of \( V \) over \( \lambda \), is defined over \( k_0' \) and is nonempty and \( \lambda \)-simple.

Let \( \Gamma \) be the graph of \( \varphi: V \to V \). Then \( \overline{\Gamma} \), the specialization of \( \Gamma \), is the graph of the endomorphism \( \overline{\varphi}: \overline{V} \to \overline{V} \). Note that \( \overline{\varphi} \) is injective again. For if \( \xi \) is any point of \( \overline{V} \) over \( K' \), choose a point \( x \) of \( V \) so that \( \xi \) is a specialization of \( x \) by [12, Lemma 3]. Then

\[
(\overline{V} \times \{\xi\}) \cdot \overline{\Gamma} = (V \times \{x\}) \cdot \Gamma
\]

is either empty or consists of one point only. Assume that we have established the surjectivity of \( \overline{\varphi} \). Then, again by (3), we find that \( (V \times \{x\}) \cdot \Gamma \) is not empty; therefore, \( \varphi \) is onto.
To prove the surjectivity of $\varphi$, it suffices to prove the surjectivity of $\varphi|_{\overline{V}(k_0')}$, where $\overline{V}(k_0')$ is the set of points on $\overline{V}$, whose coordinates are algebraic over $k_0'$. Now let $k_1'$ be any finite extension field of $k_0'$ and $\overline{V}(k_1')$ the set of points on $\overline{V}$, whose coordinates are in $k_1'$. Since $\varphi: \overline{V}(k_1') \to \overline{V}(k_1')$ is an injective map of a nonempty finite set into itself, it is onto. Hence $\varphi|_{\overline{V}(k_0')}$ is surjective.

References


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