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SOME STUDIES ON Π-COHERENT RINGS

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ABSTRACT. An internal description and a classification of Π-coherent rings are obtained by using W-ideals and FGT-projective dimension, respectively, as defined in this paper.

1. INTRODUCTION AND NOTATION

Let $\Pi = \prod R_R$ be an arbitrary product of copies of R_R. We say that R is right Π-coherent if every finitely generated submodule of Π is finitely presented (see [1]). This notion is called right strong coherence in [2] and generalized Noetherian in [3]. Many characterizations of this notion were given by the concept of module; see, for example, [1–3]. In this paper, we introduce W-ideals and FGT-projective dimension of a ring in order to give an internal description (i.e., a description by using ideals of a ring) and a classification of Π-coherent rings; meanwhile, we also get other interesting results.

All rings have a unity and all modules considered are unitary. As abbreviation, we use $f \cdot g$ for finitely generated and $f \cdot p$ for finitely presented.

For the sake of completeness let us state the following definitions.

Definition 1. Let K be a submodule of left (or right) R-module A; K is called a closed submodule if A/K is torsionless (see [9, 4.2]).

Definition 2. A ring R is called a D-ring if $I = \mathrm{rl}(I)$ for every right ideal I and $L = \mathrm{lr}(L)$ for every left ideal L of R (see [4]).

Clearly, R is a D-ring iff R/I is torsionless for every one-sided ideal I of R. In spite of this obvious description, how to give a module-theoretic characterization of a D-ring is still an interesting question.

2. W-IDEALS AND Π-COHERENT RINGS

Definition 3. A right (left) ideal I of R is called a W-right ideal if I is isomorphic to a quotient module of the dual module of an $f \cdot g$ left (right) R-module.

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For example, the right (left) annihilator ideal of a left (right) ideal I is a W-right (left) ideal by $r(I)$ $(l(I)) \cong \mathrm{Hom}_R(R/I, R)$. Hence, in D-rings every right (left) ideal is a W-right (left) ideal; moreover, every closed left (right) ideal in any ring is a W-left (right) ideal. We will prove that every $f \cdot g$ left (right) ideal is a W-left (right) ideal in any ring.

Theorem 1. (i) *R is a right Π-coherent ring iff every W-right ideal is $f \cdot g$.*

(ii) *R is right Noetherian iff R is right Π-coherent and every right ideal is a W-right ideal.*

(iii) *R is right coherent iff every W-right ideal is $f \cdot p$ if it is $f \cdot g$.*

Proof. (i) Let R be a right Π-coherent ring and I be a W-right ideal. Then there exists an $f \cdot g$ left R-module A such that $I \cong A^*/K$; but by [1, Theorem 1] A^* is $f \cdot g$, so I must be $f \cdot g$.

Conversely, let A be any $f \cdot g$ left R-module and the minimal number of generators of A be n. If $n = 1$, then there exists a left ideal I such that $A^* \cong \mathrm{Hom}_R(R/I, R) \cong r(I)$, so A^* must be $f \cdot g$ by the given condition. Next, let A' be a nonzero cyclic submodule of A. From the exact sequence $0 \to A' \to A \to A/A' \to 0$, we have the exact sequence $0 \to (A/A')^* \to A^* \xrightarrow{\Phi} A'^*$. Clearly, $A'^* \cong \mathrm{Hom}_R(R/J, R) \cong r(J)$. Here J is a left ideal of R, so $\mathrm{Im}\,\Phi$ is a W-right ideal of R and $\mathrm{Im}\,\Phi$ is $f \cdot g$; but by inductive hypothesis, $(A/A')^*$ is $f \cdot g$ and hence A^* must be $f \cdot g$. It follows from [1] that R is right Π-coherent.

(ii) Let R be right Noetherian. Then every right ideal I of R is $f \cdot g$; thus there exists an exact sequence $0 \to K \to R_R^{(n)} \to I_R \to 0$, then I_R must be a W-right ideal by $I_R \cong R_R^{(n)}/K = F^*/K$, $F^* = \mathrm{Hom}_R({}_RR^{(n)}, R)$.

Conversely, let I be any right ideal. Then I is a W-right ideal; but R is right Π-coherent; so I is $f \cdot g$ by (i), i.e., R is right Noetherian.

(iii) It holds clearly.

Remarks. (1) It follows from the proof of Theorem 1 that every one-sided ideal $f \cdot g$ is a one-sided W-ideal; the weakest condition of the converse holding is that R is a corresponding one-sided Π-coherent ring.

(2) Also from the proof of Theorem 1, it is easy to obtain that if R is left self-injective and the right annihilator of every left ideal of R is $f \cdot g$, then R is right Π-coherent.

(3) We know that right Noetherian rings $\Rightarrow$ right Π-coherent rings $\Rightarrow$ right coherent rings, but the converses do not hold generally. Many authors have discussed conditions under which coherent rings become Noetherian, but many of them are not necessary. We give the following:

Proposition 1. (i) *A right Π-coherent ring R is right Noetherian iff every right ideal is a W-right ideal.*

(ii) *A right coherent ring is right Noetherian iff every $f \cdot g$ right R-module can be embedded into an $f \cdot p$ right R-module.*

(iii) *A left and right coherent ring is left and right Π-coherent iff every $f \cdot g$ torsionless left and right R-module can be embedded in an $f \cdot g$ free module.*

Proof. (i) It holds clearly by Theorem 1.

(ii) It suffices to show the sufficiency because every $f \cdot g$ right R-module is $f \cdot p$ in right Noetherian rings. Let A be an $f \cdot g$ right R-module. Then there exists an $f \cdot p$ right R-module P such that $0 \to A \to P$ is exact; by R being right coherent, P must be a right coherent module (see [8]). Hence A is $f \cdot p$, so every $f \cdot g$ right R-module is $f \cdot p$ and R is right Noetherian.

(iii) *Sufficiency.* Let A be an $f \cdot g$ torsionless right R-module. Then there exists an $f \cdot g$ free right R-module F such that $0 \to A \to F$ is exact; but R is right coherent, so A is $f \cdot p$, i.e., R is right Π-coherent. Similarly, we can prove that R is left Π-coherent.

Necessity. Let R be right Π-coherent and A be an $f \cdot g$ torsionless left R-module. Then A^* is $f \cdot g$ by [1], so there exists an $f \cdot g$ free right R-module F such that $F \to A^* \to 0$ is exact, but A is torsionless, so $0 \to A \to F^*$ is exact; symmetrically, we can prove the right case.

For the time being, we do not know whether a left Π-coherent ring is right Π-coherent, but we can give the following.

Proposition 2. *If R is a left and right coherent ring, then R is left Π-coherent iff R is right Π-coherent.*

Proof. Suppose R is left Π-coherent, and let A be any $f \cdot g$ torsionless right R-module. Then there exists an $f \cdot g$ free right R-module F such that $0 \to A \to F$ is exact by the proof of Proposition 1(iii), but R is right coherent, so A is $f \cdot p$ and hence R is right Π-coherent; similarly, we can prove the converse.

Remark 4. We know that if R is right Noetherian, so is R/I for every ideal I and also that if R is right coherent, so is R/I for any $f \cdot g$ ideal I. For right Π-coherent rings, we have:

Proposition 3. *Let R be a right Π-coherent ring and I be an ideal that is right closed. Then R/I also is right Π-coherent.*

Proof. Let M be a $f \cdot g$ torsionless right R/I-module. Then there exists an exact sequence $0 \to M \to \Pi(R/I)_{R/I}$, but I is right closed, so there exists an exact sequence as R-module $0 \to R/I \to \Pi R_R$. Hence M is an $f \cdot g$ torsionless right R-module and so is an $f \cdot p$ right R-module; but I must be $f \cdot g$ since R is right Π-coherent, therefore, M is an $f \cdot p$ right R/I-module, i.e., R/I is right Π-coherent.

3. FGT-DIMENSIONS AND Π-COHERENT RINGS

Definition 4. A right R-module M is called right FGT-projective if $\mathrm{Ext}^1_R(M, A) = 0$ for every $f \cdot g$ torsionless right R-module A. We call $\inf\{n: \mathrm{Ext}^{n+1}_R(M, A) = 0$ for every $f \cdot g$ torsionless right R-module $A\}$ the right FGT-projective dimension of M, denoted by rFGT-proj.dim M, and sup{rFGT-proj.dim M_R for any M_R} the right FGT-projective dimension of R, denoted by rFGT-P.dim R; if the sup does not exist, we say rFGT-P.dim R infinite. Similarly, we can define the left case.

Proposition 4. *Let R be a right Π-coherent ring. Then an $f \cdot g$ torsionless right R-module is FGT-projective iff it is projective. In particular, R is right semihereditary iff it is right coherent and every $f \cdot g$ right ideal is FGT-projective.*

Proof. Let A be an $f \cdot g$ torsionless FGT-projective right R-module. Then there exists an exact sequence $0 \to K \to R^{(n)} \xrightarrow{f} A \to 0$ and K is a $f \cdot g$ torsionless right R-module; but for A FGT-projective we have the exact sequence $0 \to \mathrm{Hom}_R(A, K) \to \mathrm{Hom}_R(A, R^{(n)}) \to \mathrm{Hom}_R(A, A) \to 0$, i.e., f splits, so A is projective; the converse holds clearly.

Now let us prove the second part. By the definition of right semihereditary, the necessity holds clearly. Conversely, let I be any $f \cdot g$ right ideal. If it is FGT-projective, by using the proof of the first paragraph, we can prove that I is projective, so R is right semihereditary.

Theorem 2. $\mathrm{rFGT\text{-}P.dim}\, R = 0$ *iff* R *is a regular right self-injective ring.*

Proof. ($\Rightarrow$) By $\mathrm{rFGT\text{-}P.dim}\, R = 0$ we know that every $f \cdot g$ torsionless right R-module is injective; R immediately is a regular right self-injective ring.

($\Leftarrow$) First, we prove that R is a left Π-coherent ring. Let M be an $f \cdot g$ right R-module and the minimal number of the generators be n. If $n = 1$, there must exist a right ideal I such that $M \cong R/I$ and then $M^* \cong \mathrm{Hom}_R(R/I, R) \cong l(I)$; but R is regular right self-injective, so R is a Baer ring and $M^* \cong l(I)$ must be $f \cdot g$. Now let M_1 be a nonzero submodule of M and the minimal number of its generators be less than n. Then we have the exact sequence $0 \to \mathrm{Hom}_R(M/M_1, R) \to \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M_1, R) \to 0$ for R is right self-injective. By the induction hypothesis, $\mathrm{Hom}_R(M/M_1, R)$ and $\mathrm{Hom}_R(M_1, R)$ are $f \cdot g$, thus so is M^*, i.e., R is left Π-coherent by [1].

Next we prove that every $f \cdot g$ torsionless right R-module is projective. Since R is regular, R must be right semihereditary. By Theorem 2.11 in [2], every $f \cdot g$ torsionless right R-module is projective, but R is right self-injective. Clearly, $\mathrm{rFGT\text{-}P.dim}\, R = 0$.

Because of this theorem, we may regard the FGT-projective dimension of a ring as a measure of how far away it is from being a regular self-injective ring.

Theorem 3. *Let* R *be a right* Π*-coherent ring. Then* $\mathrm{rFGT\text{-}P.dim}\, R = n < \infty$ *iff the injective dimensions of the closed submodules of* $f \cdot g$ *free right* R*-modules are not more than* n *and* $\mathrm{Id}\, R_R = n$.

Proof. Let $\mathrm{Id}\, R_R = k$, then $\mathrm{Id}\, F_R = k$ for every $f \cdot g$ free right R-module F_R. By $\mathrm{rFGT\text{-}P.dim}\, R = n$, we know that there exists an $f \cdot g$ torsionless right R-module A such that $\mathrm{Id}\, A_R = n$. For A_R, we have the exact sequence $0 \to K \to F \to A \to 0$. Clearly, K is an $f \cdot g$ closed submodule of F. If $n > k$, we have $\mathrm{Id}\, K_R = n + 1$ by [7]; this contradicts that $\mathrm{rFGT\text{-}P.dim}\, R = n$, so $\mathrm{rFGT\text{-}P.dim}\, R = \mathrm{Id}\, R_R = n$. The other part holds clearly.

Conversely, let A be any $f \cdot g$ torsionless right R-module. By the exact sequence $0 \to K \to F \to A \to 0$ and the given condition, $\mathrm{Id}\, A_R \le n$, but $\mathrm{Id}\, R_R = n$, so $\mathrm{rFGT\text{-}P.dim}\, R = n$.

Corollary. *Let* R *be a right* Π*-coherent right. Then* R *belongs to only one of the following:*

(i) R *is regular two-sided* Π*-coherent and right self-injective;*
(ii) $0 < \mathrm{Id}\, R_R < \infty$, *and the injective dimensions of the closed submodules of any* $f \cdot g$ *free right* R*-module are not more than* $\mathrm{Id}\, R_R$;
(iii) R *is right self-injective and* $\mathrm{gl.w.dim}\, R < \infty$;
(iv) R *is regular two-sided* Π*-coherent and* $\mathrm{Id}\, R_R = \infty$;
(v) R *is not regular and is not right self-injective, but there exists at least one* $f \cdot g$ *torsionless right* R*-module, which has infinite injective dimension.*

Proof. By Theorems 2 and 3, it suffices to discuss the case of rFGT-P.dim $R = \infty$.

Since rFGT-P.dim $R = \infty$, R is not regular right self-injective by Theorem 2; thus R may be right self-injective but not regular. By [6] R is an IF ring, so gl.w.dim $R = \infty$. On the other hand, if $0 <$ rFGT-P.dim $R < \infty$, then so is $\mathrm{Id}\, R_R$ by Theorem 3. This contradicts that R is right self-injective, hence rFGT-P.dim $R = \infty$, so much for (iii).

Similarly, R may be regular right Π-coherent but not right self-injective. In this case, R is left and right semihereditary. By the proof of Theorem 2, we know that R is left Π-coherent, so every $f \cdot g$ torsionless right R-module is projective by [2, Theorem 2.11]; hence $\mathrm{Id}\, A_R \leq \mathrm{Id}\, R_R$ for any $f \cdot g$ torsionless right R-module A, but rFGT-P.dim $R = \infty$, so $\mathrm{Id}\, R_R = \infty$. Conversely, if $\mathrm{Id}\, R_R = \infty$, then rFGT-P.dim $R = \infty$. This is the case (iv).

The last case holds clearly.

Generally speaking,

$$\text{rgl.dim}\, R \neq \sup\{\mathrm{Id}\, A_R,\ \forall f \cdot g \text{ right } R-\text{module } A_R\},$$

but in Noetherian rings, the equality holds. About right Π-coherent rings, we give the following.

Proposition 5. *Let R be a right Π-coherent D-ring (see Definition 2). Then* rFGT-P.dim $R =$ rgl.dim R.

Proof. First, we prove that R must be a QF ring. Let I be any right ideal of R, then $r(l(I)) \cong \mathrm{Hom}_R(R/l(I), R)$ is $f \cdot g$ by [1], but R is a D-ring, so $I = r(l(I))$ is $f \cdot g$, i.e., R is right Noetherian. By [4, Lemma 3.1; 5, Chapter xiv, Proposition 2.1], we know that R is right self-injective, so R is a QF ring.

Clearly, every $f \cdot g$ right R-module is torsionless. According to the above mentioned, we have our equality.

Remark 5. From the proof of Proposition 5, we know that a right Π-coherent ring is a QF ring iff is a D-ring.

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