VANISHING THEOREMS FOR SINGULAR VARIETIES

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Abstract. We give generalizations of Ramanujam’s version of the Kodaira Vanishing Theorem to singular varieties.

In this note, we give generalizations of the Kodaira Vanishing Theorem to singular varieties. Let $X$ be an $n$-dimensional Cohen-Macaulay projective variety over $\mathbb{C}$ and $L$ be a line bundle on $X$. For $s \in V = H^0(X, L^m)$, $s \neq 0$, let $D_s$ be the zero-scheme of $s$, a Cartier divisor on $X$. Let $B = \bigcap_{s \in V - \{0\}} D_s$ be the base locus of the linear system $V$. Then, there is a rational map $\phi_{L^m} : X \to \mathbb{P}(V^*)$ defined outside $B$ by sending $x$ to the hyperplane of divisors containing $x$.

Theorem 1. Let $X$, $L$, and $\phi_{L^m}$ be as above and $B$ be the base locus of the linear system $H^0(X, L^m)$. Suppose that the dimension of the image of $X$ under the map $\phi_{L^m}$ is at least $k$. Then

$$H^i(X, \omega_X \otimes L) = 0 \quad \text{for } i > \max(n - k, \dim B, \dim \text{Sing}(X)),$$

where $\text{Sing}(X)$ denotes the singular locus of $X$.

Remark 1. When $X$ is a nonsingular projective variety over $\mathbb{C}$, this was proved by Ramanujam [R, Theorem 3].

Proof. First of all, we reduce the case to $m = 1$ using a finite covering of $X$. By Bertini’s Theorem, there is a basis $\{s_0, \ldots, s_r\}$ of the complete linear system $H^0(X, L^m)$ such that the zero-scheme $D_{s_i}$ of $s_i$ is smooth outside the union of $B$ and $\text{Sing}(X)$ for $i = 0, \ldots, r$. Now we construct a tower of cyclic coverings inductively. We set $Y_{-1} = X$ and $\tau_{-1}$ to be the identity map on $X$. The section $\tau_{j-1}^*(s_j)$ of $\tau_{j-1}^*(L^m)$ defines a structure of an $\mathcal{O}_{Y_{j-1}}$-algebra on $\bigoplus_{i=0}^{m-1} \tau_{j-1}^* L^{-i}$. The natural map

$$\sigma_j : Y_j = \text{Spec}_{Y_{j-1}} \left( \bigoplus_{i=0}^{m-1} \tau_{j-1}^* L^{-i} \right) \to Y_{j-1}$$

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gives an $m$-fold cyclic covering map of $Y_{j-1}$. Let
\[ \tau_j = \sigma_j \circ \tau_{j-1} : Y_j \to X \]
be the composition map and $\tau = \tau_r$. Then $Y_r$ will be smooth outside the
inverse image $\tau^{-1}(B \cup \text{Sing}(X))$. In particular, the maximum of the dimensions
of $B$ and $\text{Sing}(X)$ will be preserved under the map $\tau$. Also note that $\tau^*L$
possesses sections $t_0, \ldots, t_r$ such that $t^m = \tau^*(s_i)$ in $\tau^*L^m$. Thus the field
$\mathbb{C}(t_1/t_0, \ldots, t_r/t_0)$ is algebraic over the function field $\mathbb{C}(s_1/s_0, \ldots, s_r/s_0)$ of
$\phi_{L^n}(X)$. Hence the dimension of the image of $Y_r$ under the map $\phi_{\tau^*L}$ is
at least $k$ since the function field $K(\phi_{\tau^*L}(Y_r))$ contains $\mathbb{C}(t_1/t_0, \ldots, t_r/t_0)$.
Furthermore, we have
\[ H^i(X, \tau_*\tau^*L^{-1}) = H^i(Y_r, \tau^*L^{-1}), \]
since $\tau$ is affine. The trace map gives a splitting of the natural homomorphism
$L^{-1} \to \tau_*\tau^*L^{-1}$, and hence $L^{-1}$ is a direct summand of $\tau_*\tau^*L^{-1}$.
Moreover, $Y_r$ is Cohen-Macaulay since $\tau$ is flat. Now by Serre duality,
$H^i(Y_r, \omega_{Y_r} \otimes \tau^*L) = 0$ will imply that $H^i(X, \omega_X \otimes L) = 0$. Hence we may
assume $m = 1$.

Let $B$ be the base locus of the linear system $V = H^0(X, L)$. Let $\pi : \tilde{X} \to X$
be a desingularization of the blow-up of $X$ along $B$. There is a morphism
\[ \psi : \tilde{X} \to \mathbb{P}(V^*) \]
extending the morphism $\phi_L : X - B \to \mathbb{P}(V^*)$. Then we have
\[ \psi^*\mathcal{O}(1) = \pi^*L \otimes \mathcal{O}(-E) \]
where $E$ is an effective divisor on $\tilde{X}$ with support in $\pi^{-1}(B)$. Thus there is
an exact sequence
\[ 0 \to \psi^*\mathcal{O}(1) \otimes \omega_{\tilde{X}} \to \pi^*L \otimes \omega_{\tilde{X}} \to \pi^*L \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E \to 0. \]
We claim that
\[ R^q\pi_*(\psi^*\mathcal{O}(1) \otimes \omega_{\tilde{X}}) = 0 \quad \text{for } q > 0. \]
Let $H$ be a smooth divisor of a section of $\psi^*\mathcal{O}(1)$. Consider the Poincaré
residue sequence
\[ 0 \to \omega_{\tilde{X}} \to \mathcal{O}(H) \otimes \omega_{\tilde{X}} \to \omega_{H} \to 0. \]
Grauert-Riemenschneider Vanishing Theorem [GR, Satz 2.3] implies that
\[ R^q\pi_*\omega_{\tilde{X}} = R^q\pi_*\omega_{H} = 0 \quad \text{for } q > 0. \]
Thus we conclude that
\[ R^q\pi_*(\psi^*\mathcal{O}(1) \otimes \omega_{\tilde{X}}) = R^q\pi_*(\mathcal{O}(H) \otimes \omega_{\tilde{X}}) = 0 \quad \text{for } q > 0. \]
Hence we get from (1)
\[ 0 \to \pi_*(\psi^*\mathcal{O}(1) \otimes \omega_{\tilde{X}}) \to \pi_*(\pi^*L \otimes \omega_{\tilde{X}}) \to \pi_*(\pi^*L \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E) \to 0. \]
Since $\pi_*(\pi^*L \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E)$ is supported on $B$, there is a surjection
\[ H^i(X, \pi_*(\psi^*\mathcal{O}(1) \otimes \omega_{\tilde{X}})) \to H^i(X, \pi_*(\pi^*L \otimes \omega_{\tilde{X}})) \quad \text{for } i > \dim B. \]
(In fact, this is an isomorphism for $i > \dim B + 1$.) Via the Leray spectral sequence, this induces a surjection

$$H^i(\widetilde{X}, \psi^*\mathcal{O}(1) \otimes \omega_{\widetilde{X}}) \to H^i(\widetilde{X}, \pi^*L \otimes \omega_{\widetilde{X}}) \quad \text{for} \ i > \dim B,$$

since $R^q\pi_* (\pi^*L \otimes \omega_{\widetilde{X}}) = L \otimes R^q\pi_* \omega_{\widetilde{X}} = 0$, $q > 0$, by the Grauert-Riemenschneider Vanishing Theorem [GR, Satz 2.3].

On the other hand, we have

$$H^i(\widetilde{X}, \psi^*\mathcal{O}(1) \otimes \omega_{\widetilde{X}}) = 0 \quad \text{for} \ i > n - k,$$

by Ramanujam's generalization of the Kodaira Vanishing Theorem [R, Theorem 3]. Therefore, we have

$$H^i(\widetilde{X}, \pi^*L \otimes \omega_{\widetilde{X}}) = 0 \quad \text{for} \ i > \max(n - k, \dim B).$$

Again from the Leray spectral sequence and the Grauert-Riemenschneider Vanishing Theorem [GR, Satz 2.3], it follows that

$$H^i(X, L \otimes \pi_* \omega_{\widetilde{X}}) = 0 \quad \text{for} \ i > \max(n - k, \dim B).$$

Now our proof follows from an exact sequence

$$0 \to \pi_* \omega_{\widetilde{X}} \to \omega_X \to Q \to 0$$

where $Q$ is supported on $\text{Sing}(X)$.

**Corollary 2.** Let $X$ be an $n$-dimensional projective variety over $\mathbb{C}$ with only rational singularities outside the base locus $B$ of $H^0(X, L^m)$. Suppose that the dimension of the image of $X$ under the map $\phi_{L^m}$ is at least $k$. Then

$$H^i(X, \omega_X \otimes L) = 0 \quad \text{for} \ i > \max(n - k, \dim B).$$

**Proof.** First, we will show that the reduction step to $m = 1$ works. Let $\pi : \widetilde{X} \to X$ be a desingularization of the blow-up of $X$ along $B$. By Bertini's Theorem, there is a basis $\{s_0, \ldots, s_r\}$ of $H^0(X, L^m)$ such that the zero schemes of $\pi^*(s_i)$ are of the form $H_i + E$ where $H_i$ are smooth hypersurfaces and $E$ is supported on $\pi^{-1}(B)$. Let $\tau : Y_r \to X$ be a branched covering of $X$ over the union of the divisors $D_i$ constructed as in the proof of the theorem. Consider the fibre product:

$$\begin{array}{ccc}
Y_r & \xrightarrow{\tau} & X \\
\downarrow \rho & & \downarrow \pi \\
Y_r & \xrightarrow{\tau} & X
\end{array}$$

Since cohomology commutes with flat base extension [H, III, 9.3], there are natural isomorphisms

$$R^i\rho_* \mathcal{O}_{\widetilde{Y}_r} \cong \tau^* R^i\pi_* \mathcal{O}_{\widetilde{X}}.$$

Since $X$ has only rational singularities outside $B$, we have that $R^i\rho_* \mathcal{O}_{\widetilde{Y}_r}$ is supported on $\tau^{-1}(B)$ when $i > 0$ and $\rho_* \mathcal{O}_{\widetilde{Y}_r} = \mathcal{O}_{Y_r}$ on $Y_r - \tau^{-1}(B)$. Moreover, $\widetilde{Y}_r$ is smooth outside $\tau^{-1}(E)$ since the zero schemes of $\pi^*(s_i)$ are smooth on $X - E$. Thus $Y_r$ has only rational singularities outside $\tau^{-1}(B)$. Note that the dimension of $B$ is unchanged under the map $\tau$. Hence we may assume $m = 1$. 

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From [K, Proposition, p. 50], we obtain an exact sequence

(6) \[ 0 \to \pi_* \omega_X \to \omega_X \to Q \to 0 \]

where \( Q \) is supported on the base locus \( B \). The corollary follows from (4) and this sequence. \( \square \)

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