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MINIMAL SURFACES AND H-SURFACES IN NONPOSITIVELY CURVED SPACE FORMS

BENNETT PALMER

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ABSTRACT. We show that if the Gauss curvature of a surface of constant mean curvature in a nonpositively curved space form is sufficiently pinched, the surface is stable. In this case, we also give an upper bound for the inradius. We then show that the inradius of a stable minimal surface in Euclidean space, which is contained in a solid cylinder, is bounded above by a constant depending only on the radius of the cylinder.

Let $M^3(c)$ denote a 3-dimensional oriented space form of constant sectional curvature $c \leq 0$. Let $X: M \to M^3(c)$ be a smooth immersion of a smoothly bounded surface M with curvature K and mean curvature h. Set

$$\overline{K} = \max_M K, \qquad \underline{K} = \min_M K.$$

We show

Theorem I. *For $h, c \in \mathbb{R}$ with $-A^2 := h^2 + c \leq 0$, there exist universal constants $\omega(c, h) \geq e^2$ with the following property:*

If $M \subset M^3(c)$ is a smooth orientable surface with constant mean curvature h and

$$(-A^2 - \underline{K})/(-A^2 - \overline{K}) \leq \omega(c, h)$$

then M is stable. In addition, $7.4\ldots = e^2 \leq \omega(0, 0) \leq 10.75\ldots$ holds.

Theorem II. *If $-\infty < \underline{K}, \overline{K} < -A^2$, and M contains a geodesic ball $B_r(x_0)$ of radius r then*

$$r^2 \leq \frac{\pi^2}{4(-A^2 - \overline{K})} \log_e[(-A^2 - \underline{K})/(-A^2 - \overline{K})].$$

In the second part of this paper we consider surfaces in $\mathbb{E}^N$ which are extrinsically bounded in some way. Let $C_R = \{X = (X_1, X_2, X_3) \in \mathbb{E}^3 \mid x_1^2 + x_2^2 < R^2\}$. We show

Theorem III. *There exists a constant $c_1 > 0$ with the following property: If $M \subset \mathbb{E}^3$ is an orientable stable minimal surface with $B_r(p) \subset M$ and $M \subset C_{R_1} \backslash C_{R_2}$ for some $R_1 > R_2 \geq 0$ then $r^2 \leq c_1(R_1^2 - R_2^2)$.*

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Before beginning the proofs, we review some basic facts about surfaces in $M^3(c)$.

Let $M \to M^3(c)$ be an orientable surface. Let ds^2 denote the induced metric. (M, ds^2) may be considered a Riemann surface in a natural way by introducing isothermal coordinates (x, y) and using $z = x + iy$ as a complex coordinate. Doing so, ds^2 may be expressed as $ds^2 = e^\rho |dz|^2$ and the curvature is

$$K = -2e^{-\rho}\rho_{z\bar{z}}. \tag{1}$$

The second fundamental form of M has an expression

$$\Pi = \mathrm{Re}\{\phi dz^2 + he^\rho dz\, d\bar{z}\}$$

where ϕdz^2 is an invariant quadratic differential and h is the mean curvature. The fundamental equations of the immersion are those of Gauss

$$|\phi|^2 = e^{2\rho}(h^2 + c - K) \tag{2}$$

and Codazzi

$$\phi_{\bar{z}} = e^\rho h_z. \tag{3}$$

When $h = \text{const}$, (3) implies that ϕ is holomorphic in z. It then follows that either $\phi \equiv 0$ and M is totally umbilic or the zeros of ϕ are isolated.

Lemma 1. *Let $M \subset M^3(c)$, $c \leq 0$, have constant mean curvature h such that $h^2 + c \leq 0$. Assume M has no umbilics. Then the conformal metric*

$$d\tilde{s}^2 \equiv (h^2 + c - K)ds^2 \tag{4}$$

has curvature $\widetilde{K}$ satisfying

$$\widetilde{K} \geq 1. \tag{5}$$

Proof. Using (1)–(3) one has

$$\begin{aligned} 0 = \Delta \log|\phi| &= \Delta \log(-A^2 - K)^{1/2} + \Delta\rho \\ &= \Delta \log(-A^2 - K)^{1/2} - 2K \end{aligned} \tag{6}$$

where $\Delta = 4e^{-\rho}\partial z \partial \bar{z}$. Therefore, using (1) to compute $\widetilde{K}$ one has

$$\widetilde{K} = -(-A^2 - K)^{-1}\{\Delta \log(-A^2 - K)^{1/2} - K\} = (-A^2 - K)^{-1}(-K).$$

Since $K < 0$ on M, (5) follows.

Proposition 1. *Again assume $h = \text{const}$, $h^2 + c \equiv -A^2 \leq 0$, and*

$$0 \leq -A^2 - \overline{K} \leq -A^2 - K \leq -A^2 - \underline{K} < \infty. \tag{7}$$

Then the first (nontrivial) eigenvalue λ_1 of the problem

$$\begin{cases} \Delta\psi + 2\lambda(-A^2 - K)\psi = 0 & \text{on } M, \\ \psi = 0 & \text{on } \partial M \end{cases} \tag{$*$}$$

satisfies

$$[\tfrac{1}{2}\log((-A^2 - \underline{K})/(-A^2 - \overline{K}))]^{-1} \leq \lambda_1. \tag{8}$$

Proof. Let $\psi \geq 0$ be the eigenfunction corresponding to λ_1. Let $g(x, y)$ denote the positive Green's function of M. Then

$$\begin{aligned}\psi(x) &= \lambda_1 \int_\Omega (-2A^2 - 2K(y))\psi(y) g(x, y) * 1(y) \\ &\leq \lambda_1 \int_\Omega -2K(y)\psi(y) g(x, y) * 1(y).\end{aligned}$$

Therefore,

$$|\psi(y)| \leq \lambda_1 \|\psi\|_\infty \int -2K(y) g(x, y) * 1(y) \equiv \lambda_1 \|\psi\|_\infty \nu(x),$$

where ν solves

$$\begin{aligned}\Delta\nu &= 2K \quad \text{in } M, \\ \nu &\equiv 0 \quad \text{on } \partial M.\end{aligned} \tag{9}$$

Choosing x where ψ achieves its maximum we arrive at

$$1 \leq \lambda_1 \|\nu\|_\infty. \tag{10}$$

By (6) and (9)

$$\Delta(\nu - \log(-A^2 - K)^{1/2}) = 0 \quad \text{in } M$$

and on ∂M

$$\nu - \log(-A^2 - K)^{1/2} \leq -\log(-A^2 - \overline{K})^{1/2}.$$

It follows from the maximum principle and (8) that on M

$$\begin{aligned}\nu &\leq \log(-A^2 - K)^{1/2} - \log(-A^2 - \overline{K})^{1/2} \\ &\leq \log(-A^2 - \underline{K})^{1/2} - \log(-A^2 - \overline{K})^{1/2}.\end{aligned} \tag{11}$$

Using this and (10), (8) follows.

Proof of Theorem I. The surface M is stationary for the functional

$$J = \text{area} + 2H(\text{enclosed 3-volume}).$$

The second variation of J for variations of the form $\psi \cdot N$ where N is the unit normal to M and $\psi \in C_0^\infty(M)$ is given by $\delta^2 J = \int -\psi L\psi$. Here L is the selfadjoint elliptic operator $L\psi = \Delta\psi + 2(-2A^2 - K)\psi$. Assuming the hypothesis of the theorem, we have by Proposition 1, $\lambda_1 \geq 1$. Using integration by parts we obtain

$$\begin{aligned}\delta^2 J &= \int -\psi L\psi = \int (|\nabla\psi|^2 - 2(-2A^2 - K)\psi^2) \\ &\geq \int (|\nabla\psi|^2 - 2(-A^2 - K)\psi^2) \geq 0\end{aligned}$$

and M is stable.

The upper bound for $\omega(0, 0)$ follows from Example I.

Remark. We have shown that under the hypothesis of Theorem I, the second variation of J is nonnegative for all compactly supported variations. When $h \neq 0$ this is stronger than the condition that $\delta^2 J$ be nonnegative for all volume-preserving variations (cf. [B-DC]).

Example I. Let $C \subset \mathbb{R}^3 \approx \mathbb{C} \times \mathbb{R}$ be the catenoid parameterized by $X(u, v) = (e^{iv}\cosh(u), u)$, $(u, v) \in \mathbb{R} \times [0, 2\pi)$. One easily computes that the curvature is given by $K = -(\cosh u)^{-4}$ and that the support function is given by $s = -1 + u\tanh(u)$.

Denote by Ω_t the symmetric "waist" domain of the catenoid given by $|u| < t$. Then Ω_t will be stable as long as s is negative, that is, for $t < u_1 \approx 1.2\ldots$. For Ω_t

$$e^2 \geq \underline{K}/\overline{K} = (\cosh t)^4$$

holds for $t < t_1 \approx 1.0850\ldots$. It follows that Theorem I has correctly predicted stability in this case. Furthermore, for Ωu_1, $\underline{K}/\overline{K} \approx 10.75\ldots$ furnishes the upper bound in the corollary. Finally the total curvature of Ω_t is

$$2\pi \int_{-t_1}^{t_1} \cosh^{-2} u\, du \approx 2\pi(1.590\ldots) > 2\pi.$$

Consequently the criteria of Theorem I is independent of the Barbosa-DoCarmo result [B-DC1].

To prove Theorem II we state without proof a special case of an eigenvalue estimate due to Gage [G].

Theorem (Gage). *Let $\widetilde{B}_{\tilde{r}}$ be a geodesic ball of radius $\tilde{r}$ contained in a surface of curvature $\widetilde{K} \geq -\beta^2 = \text{const}$. Then the first Dirichlet eigenvalue of the Laplacian $\widetilde{\Delta}$ on $\widetilde{B}_r$ satisfies*

$$\tilde{\lambda}_1 \leq \pi^2/\tilde{r}^2 + \beta^2/4. \tag{12}$$

Proof of Theorem II. For a region $\Omega \subset M$ let $\tilde{\lambda}_1(\Omega)$ denote the first Dirichlet eigenvalue of $\widetilde{\Delta}$ for Ω. Here $\widetilde{\Delta}$ is the Laplacian for the metric $d\tilde{s}$ of Lemma 1. Let $\lambda_1(\Omega)$ be the first eigenvalue of the problem

$$\begin{aligned} \Delta\psi + 2\lambda(-A^2 - K)\psi &= 0 \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{13}$$

Since $\widetilde{\Delta} = (-A^2 - K)^{-1}\Delta$, it is clear that $\tilde{\lambda}_1(\Omega) = 2\lambda_1(\Omega)$. Let γ be a minimizing geodesic of length $\tilde{r}$ for the metric $d\tilde{s}$. Then

$$\tilde{r} = \int_\gamma (-A^2 - K)^{1/2}\, ds \geq (-A^2 - \overline{K})^{1/2} \int_\gamma ds.$$

It follows that $\widetilde{B}_{r(-A^2-\overline{K})^{1/2}} \subset B_r$. By a well-known monotonicity property of eigenvalues

$$\tilde{\lambda}_1(\widetilde{B}_{r(-A^2-\overline{K})^{1/2}}) \geq \tilde{\lambda}_1(B_r) = 2\lambda_1(B_r).$$

So by Lemma 1 and Gage's Theorem with $\beta = 0$, we have

$$\frac{\pi^2}{r^2(-A^2 - \overline{K})} \geq 2\lambda_1(B_r).$$

Combining this with the lower bound of Proposition 1 yields the result.

We now consider surfaces in $\mathbb{E}^N$ which are extrinsically bounded.

Lemma 2. *Let $M \subset \mathbb{E}^3$ be a minimal surface. Let $\Omega \subset M$ be a smoothly bounded subdomain, and let μ_1 be the first Dirichlet eigenvalue of the Laplacian in Ω. Assume*

$$M \subset C_{R_1} \backslash C_{R_2}, \qquad R_1 > R_2 \geq 0. \tag{14}$$

Then

$$\mu_1 \geq \frac{2}{R_1^2 - R_2^2}. \tag{15}$$

Proof. Define $\tau = \frac{1}{2}(x_1^2 + x_2^2)$. Let $N = (N_1, N_2, N_3)$ be a unit normal defined on a neighborhood in M. Then

$$\begin{aligned}\Delta\tau &= \frac{1}{2}\sum_{i=1,2}(2x_i\Delta x_i + 2\|\nabla x_i\|^2) = \|\nabla x_1\|^2 + \|\nabla x_2\|^2 \\ &= 1 - N_1^2 + 1 - N_2^2 = 1 + N_3^2 \geq 1.\end{aligned}$$

By (14) we have

$$\frac{R_2^2}{2} < \tau < \frac{R_1^2}{2}. \tag{16}$$

Let $\psi \geq 0$ be a solution of $\Delta\psi + \mu\psi = 0$ with $\psi \equiv 0$ on $\partial\Omega$. Then

$$\psi(x) = \mu_1 \int_\Omega \psi(y) g(x, y) * 1(y)$$

and consequently

$$|\psi(x)| \leq \mu_1 \|\psi\|_\infty \int_\Omega g(x, y) * 1(y).$$

Taking x values where ψ achieves its maximum yields

$$1 \leq \mu_1 \max_{x\in\Omega} \int_\Omega g(x, y) * 1(y) \equiv \mu_1 \max_{x\in\Omega} S(x) \tag{17}$$

where S solves

$$\begin{aligned}\Delta S &= -1 \quad \text{in } \Omega, \\ S &\equiv 0 \quad \text{on } \partial\Omega.\end{aligned}$$

Therefore,

$$\Delta(S + \tau) \geq 0 \quad \text{in } \Omega \qquad \text{and} \qquad S + \tau = \tau \leq R_1^2/2 \quad \text{on } \partial\Omega.$$

By the maximum principle

$$S \leq \frac{R_1^2}{2} - \tau \leq \frac{R_1^2}{2} - \frac{R_2^2}{2} \quad \text{in } \Omega.$$

Combining this with (17) proves (15).

Proof of Theorem III. Since M is stable, it follows by a result of Schoen [Sc, Corollary 4] that there is an estimate $K(x) \geq -2\alpha/r^2$, for all $x \in B_{r/2}$, where α is a universal constant. Using this lower bound for K in Gage's upper bound for μ_1 gives

$$\mu_1 \leq \frac{\alpha}{4r^2} + \frac{\pi^2}{r^2} =: \frac{4C_1}{r^2}.$$

Combining this with the lower bound for μ_1 in Lemma 2 gives the result.

Corollary. *Let $M \subset \mathbb{E}^3$ be a complete minimal surface. If there exists $p \in M$ such that*

$$\limsup_{r\to\infty} \frac{1}{r} \int_{B_r(P)} (-K) = 0 \tag{18}$$

then M is not contained in a cylinder.

Proof. Assume to the contrary that $M \subset C_R$ for some R, $0 < R < \infty$. Let r_0 be a constant with $r_0^2 > c_1 R^2$ with c_1 as in Theorem III. Then any disc $B_{r_0} \subset M$ is unstable. By a result of Barbosa and DoCarmo [B-DC1] $\int_{B_{r_0}} (-K) > 2\pi$. Taking the sequence $r_n = nr_0$, one finds that since $B_{r_n}(p)$ contains at least n disjoint geodesic balls of radius r_0,

$$\frac{1}{r_n} \int_{B_{r_n}(P)} (-K) > \frac{2\pi n}{nr_0} = \frac{2\pi}{r_0} \gg 0,$$

giving a contradiction.

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Technische Universität Berlin, Str. des 17 Juni 135, 1000 Berlin 12, Germany

Current address: Science Laboritories, University of Durham, Durham, DH1-3LE, United Kingdom

E-mail address: bennett.palmer@durham.ac.uk