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FINITE LOOP SPACES WITH MAXIMAL TORI HAVE FINITE WEYL GROUPS

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ABSTRACT. A finite loop space X is said to have a *maximal torus* if there is a map $f: BT \to BX$ where T is a torus such that $\operatorname{rank}(T) = \operatorname{rank}(X)$ and the homotopy fibre of f has the homotopy type of a finite complex.

The *Weyl group* W_f of f is the set of homotopy classes $w: BT \to BT$ such that

$$\begin{array}{ccc} BT & \xrightarrow{w} & BT \\ & {}_f\searrow \quad \swarrow_f & \\ & BX & \end{array}$$

homotopy commutes. In this note we prove that W_f is always finite.

By a *finite loop space* we understand a topological space X such that:

- X has the homotopy type of a finite complex,
- X has the homotopy type of a topological group.

N.B. It is not assumed both structures may be realized on the same space simultaneously.

A compact Lie group is a finite loop space, but there are well-known examples (see, e.g., the book of Richard Kane [2] and the references there) of finite loop spaces which are not Lie groups. An important idea, introduced by Rector [7] (see also [6, I, II, III; 5]) for the study of finite loop spaces is that of a *maximal torus*.

Definition. If X is a finite loop space, a *maximal torus* for X is a map $f: BT \to BX$ where $B(-)$ is the classifying space functor and T is a torus such that

- $\operatorname{rank}(T) = \operatorname{rank}(X)$ (see next page for definition of rank),
- the homotopy fibre F of f has the homotopy type of a finite complex.

A finite loop space need not have a maximal torus as Rector showed [7], and if X is a fake Lie group, then it has a maximal torus if and only if it is a Lie group [6, I–III; 5]. For a finite loop space with maximal torus, Rector adapted a further concept from Lie theory, namely, the *Weyl group*.

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Definition. If X is a finite loop space with a maximal torus $f\colon BT \to BX$, then the *Weyl group of* f, W_f, is the group of homotopy classes of maps $w\colon BT \to BT$ such that

$$\begin{array}{ccc} BT & \xrightarrow{w} & BT \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle f} \\ & BX & \end{array}$$

homotopy commutes.

Notice that $[BT, BT] \simeq \mathrm{GL}(n; \mathbb{Z})$, where $n = \mathrm{rank}(T)$, so $W_f \leq \mathrm{GL}(n; \mathbb{Z})$. If G is a compact connected Lie group and X a fake Lie group (e.g., $X = G$) of type G (see [6, I] for definitions) with a maximal torus $f\colon BT \to BX$ (e.g., $f = B_\rho$ where $\rho\colon T \hookrightarrow G$ is the inclusion of a maximal torus), then $W_f \simeq W_G$. However, for a general finite loop space there seems no reference for the fact that W_f is a *finite* group.† We rectify this in the following:

Theorem. *Let X be a connected finite loop space and suppose X has a maximal torus $f\colon BT \to BX$. Then the Weyl group W_f is finite, and $|W_f|$ is a divisor of $d = d_1 \cdots d_n$ where $(2d_1 - 1, \dots, 2d_n - 1)$ is the type of X.*

The *type* of a finite loop space X (or H-space) is defined following Hopf (see, e.g., [3]) who showed

$$H^*(X; \mathbb{Q}) = H^*(S^{2d_1-1} \times \cdots \times S^{2d_n-1}; \mathbb{Q})$$

where the integers $2d_1 - 1, \dots, 2d_n - 1$ are called the type of X and n its rank. If X is a Lie group, then the rank as just defined coincides with its rank as a Lie group.

Proof. Let X have rank n and type $(2d_1 - 1, \dots, 2d_n - 1)$. By the Leray-Samelson theorem [3] it follows that for sufficiently large primes p or $p = 0$

$$H^*(X; \mathbb{F}_p) \simeq E(u_1, \dots, u_n)$$

where $(\mathbb{F}_p := \mathbb{Z}/p,\ \mathbb{F}_0 = \mathbb{Q})$

$$\deg u_i = 2d_i - 1, \qquad i = 1, \dots, n.$$

From the Milnor-Moore spectral sequence [4] it then follows

$$H^*(BX, \mathbb{F}_p) \simeq \mathbb{F}_p[\rho_1, \dots, \rho_n]$$

where

$$\deg \rho_i = 2d_i, \qquad i = 1, \dots, n.$$

Let F denote the fibre of the maximal torus $f\colon BT \to BX$. Consider the Serre spectral sequence $\{E_r, d_r\}$

$$E_r \Rightarrow H^*(BT; \mathbb{F}_p),$$
$$E_2 = H^*(BX; \mathbb{F}_p) \otimes H^*(F; \mathbb{F}_p).$$

Since F is finite, E_2^{**} is a finitely generated $H^*(BX; \mathbb{F}_p)$ module. If $p = 0$, or p is sufficiently large, $H^*(BX; \mathbb{F}_p)$ is a noetherian ring, and hence an easy argument shows that $H^*(BT; F_p)$ is a finitely generated $H^*(BX; \mathbb{F}_p)$ module. Recall

$$H^*(BT; \mathbb{F}_p) \simeq \mathbb{F}_p[t_1, \dots, t_n]$$

† In [8] Rector and Stasheff state that this is the case but give no proof.

where

$$\deg t_i = 2, \qquad i = 1, \dots, n.$$

Therefore,

$$f^*: H^*(BX; \mathbb{F}_p) \to H^*(BT; \mathbb{F}_p)$$

must be monic for $H^*(BT; \mathbb{F}_p)$ to be finitely generated over $H^*(BX; \mathbb{F}_p)$, and furthermore by Maccauley's theorem [9] $H^*(BT; \mathbb{F}_p)$ is a free $H^*(BX; \mathbb{F}_p)$ module.

From this point on assume that p is a prime that is sufficiently large and $p \nmid d = d_1 \cdots d_n$. Then according to Adams-Wilkerson [1] there exists an essentially unique embedding

$$\varphi: H^*(BX; \mathbb{F}_p) \hookrightarrow H^*(BT; \mathbb{F}_p)$$

and a finite group $W(p) \leq \mathrm{GL}(n; \mathbb{F}_p)$ with

$$\varphi(H^*(BX; \mathbb{F}_p)) = H^*(BT; \mathbb{F}_p)^{W(p)}$$

and

$$|W(p)| = d.$$

The uniqueness of φ allows us to suppose that $\varphi = f^*$. By the very definition of W_f we have

$$H^*(BT; \mathbb{F}_p)^{W(p)} \leq H^*(BT; \mathbb{F}_p)^{W_f}.$$

Let $FF(-)$ denote the field of fractions functor, and set $H^*(BT; \mathbb{F}_p) = B$. Then

$$FF(B)^{W(p)} = FF(B^{W(p)}) \leq FF(B^{W_f}) \leq FF(B)^{W_f}$$

where the first equality results from [6, I.3.2]. By Galois theory it follows that the image of W_f under the reduction homomorphism

$$\rho: \mathrm{GL}(n; \mathbb{Z}) \to \mathrm{GL}(n; \mathbb{F}_p)$$

is contained in $W(p)$.

Suppose that $|W_f| > d$. Let $1 = w_0, w_1, \dots, w_d \in W_f$ be $d+1$ distinct elements. Choose p even larger if necessary so that

$$p \nmid w_r(i, j) - w_s(i, j) : \begin{cases} 0 \leq r, s \leq d, \\ 1 \leq i, j \leq n. \end{cases}$$

Then the mod p reduction of these elements would have to be distinct, contrary to the fact

$$\rho(W_f) \leq W(p), \qquad |W(p)| = d.$$

Therefore, W_f is finite, and p is monic if p is large enough. □

Remark. The referee has suggested an alternate proof. In outline the idea is as follows. Work over the rationals $\mathbb{Q}$ instead of the finite field $\mathbb{Z}/p$. As above $A := H^*(BT; \mathbb{Q})$ is a finitely generated module over $B := H^*(BX; \mathbb{Q})$. (One must argue that $H^*(BX; \mathbb{Q}) \to H^*(BT; \mathbb{Q})$ is monic and:) Therefore $A \supseteq B$ is an integral ring extension. The canonical map $W \to \mathrm{Aut}(FF(A))$ of the Weyl group W into the automorphism group of the field of fractions of A is an injection. $FF(A) \supseteq FF(A)^W \supseteq FF(B)$ is a sequence of field extensions, where the first is a Galois extension. An argument using Poincaré series (e.g., T. Springer, *Invariant theory*, Lecture Notes in Math., vol. 585,

§2.5.6, Springer, New York) shows that the degree $[FF(A) : FF(B)]$ of the extension $FF(A) \supseteq FF(B)$ is given by $\prod d_i$. Hence, the order of W divides $\prod d_i$, and in particular, the order of W is finite.

N.B. The paper of Rector-Stasheff [8] was published in 1974 and the book of Springer was first published in 1977.

The preceding theorem answers yet one more question about the Weyl group of a maximal torus for a finite loop space but leaves open many more. In particular:

- Is the Weyl group of a finite loop space with maximal torus nontrivial?
- Is $H^*(BX; \mathbb{F}_p)$ the ring of invariants of W_f acting on $H^*(BT; \mathbb{F}_p)$?

Affirmative answers would classify the possible Weyl groups as the groups generated by reflections and answer in the affirmative the question of whether the type of a finite loop space with maximal torus must coincide with that of a Lie group.

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