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COMPLETE MINIMAL SURFACES AND THE PUNCTURE NUMBER PROBLEM

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ABSTRACT. Given a nonnegative integer g, let $\mathscr{P}(g)$ denote the set of integers N such that an arbitrary compact Riemann surface with genus g can be completely conformally and minimally immersed in $\mathbb{R}^3$ (with finite total curvature) with exactly N punctures. We prove that the infimum of $\mathscr{P}(g)$ is at most $4g$ and that the set $\mathscr{P}(g)$ may not miss any $3g$ consecutive integers larger than the infimum of $\mathscr{P}(g)$.

1. INTRODUCTION

A conformal immersion from a Riemann surface M to $\mathbb{R}^3$ is said to be minimal if its component functions are harmonic, and the maximum principle for harmonic functions prohibits any compact minimal surface. The tangential Gauss map of a conformal minimal immersion $f\colon M \to \mathbb{R}^3$ is the map $\Phi\colon M \to G(3, 2)$ taking $p \in M$ to the (negatively) oriented tangent plane $f_* T_p M \subset \mathbb{R}^3$. $G(3, 2)$ is the Grassmann manifold of oriented two-planes in $\mathbb{R}^3$, and it lies naturally in $\mathbb{C}P^2$. Moreover, the minimality of f implies that the map $\Phi\colon M \to \mathbb{C}P^2$ is holomorphic. A fundamental theorem of Chern and Osserman [CO] states that a complete minimal surface M (relative to the induced metric) is of finite total Gaussian curvature if and only if M is holomorphically equivalent to a compact Riemann surface M_g punctured at a finite number of points and the tangential Gauss map extends holomorphically to all of M_g. Thus the theory of complete minimal surfaces of finite total curvature is intimately linked to that of compact Riemann surfaces.

Given a nonnegative integer g, define the set $\mathscr{P}(g) \subset \mathbb{Z}^+$ by the following prescription: $N \in \mathscr{P}(g)$ if and only if any compact Riemann surface of genus g can be completely conformally and minimally immersed in $\mathbb{R}^3$ (with finite total curvature) with exactly N punctures. In less precise terms, the set $\mathscr{P}(g)$ consists of admissible puncture numbers for an arbitrary compact Riemann surface of genus g. For example, it is well known [JM] that $\mathscr{P}(0) = \mathbb{Z}^+$. Hereafter we will exclude the case $g = 0$ from our consideration. We can now state the

Puncture number problem. Determine the set $\mathscr{P}(g)$, $g > 0$.

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Our main result is the following

Theorem. (a) $4 \in \mathscr{P}(1)$,
(b) $\inf \mathscr{P}(g) \leq 4g$,
(c) $\mathscr{P}(g)$ *may not miss any* $3g$ *consecutive integers larger than* $\inf \mathscr{P}(g)$.

Note that (c) implies that $|\mathscr{P}(g)| = \infty$. That is to say, there are infinitely many admissible puncture numbers for any genus.

It should be remarked that the puncture number problem is closely related to the moduli problem for complete minimal surfaces. In fact, Xiaokang Mo recently constructed a complex space parametrizing the set of complete minimal surfaces with a fixed number of punctures using an earlier result of the author [Y1] on the puncture number problem. For details of Mo's construction see [M].

2. An estimate lemma and the proof of the Theorem

To prove the Theorem we will need the following

Lemma. *Let* M_g *denote a compact Riemann surface of genus* $g > 0$. *Also let* $F \in H^0(M_g, \mathscr{M}^*)$ *be a not identically zero meromorphic function on* M_g, *and put*

$$n = n_F = \text{ the number of distinct poles of } F,$$
$$d = d_F = \text{ the degree of } F = \text{ the total number of poles of } F,$$
$$m = m_F = \text{ the number of distinct zeros of } dF.$$

(i) *There exists a conformal minimal immersion, complete with respect to the induced metric,* $M_g \backslash \Sigma_F \to \mathbb{R}^3$, *where the puncture set* Σ_F *satisfies*

$$(*) \qquad n + m \leq |\Sigma_F| \leq 2n + d + 3g - 2.$$

(ii) *Suppose* D *is any divisor on* M_g *satisfying the following conditions*:

$$\deg D = -g, \qquad D^+ = (dF)_0, \qquad D^- \geq (dF)_\infty,$$
$$\text{support } D^- = \text{support}(dF)_\infty,$$

where $D = D^+ - D^-$ *with* $D^\pm > 0$. *Also let* $G \in L(-D)$, *and put*

$$m' = \text{ the number of distinct zeros of } G.$$

Then there exists a complete conformal minimal immersion $M_g \backslash \Sigma_{F,G} \to \mathbb{R}^3$ *with*

$$(**) \qquad |\Sigma_{F,G}| = n + m'.$$

Proof. For some $b_i \in \mathbb{Z}^+$ and distinct points $p_i \in M_g$ we have

$$(F)_\infty = \sum_{i=1}^{n} b_i p_i.$$

Then $d = \sum b_i = \deg(F)_\infty$. Consider the meromorphic 1-form dF. We have

$$(dF)_\infty = \sum (b_i + 1) p_i.$$

For some $a_j \in \mathbb{Z}^+$ and distinct points $q_j \in M_g$ we have

$$(dF)_0 = \sum_{j=1}^{m} a_j q_j .$$

Since $(dF) = (dF)_0 - (dF)_\infty$ is a canonical divisor, its degree is $2g-2$ so that

$$\sum a_j = (2g-2) + n + d .$$

Let $D = D^+ - D^- \in \mathrm{Div}(M_g)$ be as above. We then have

$$D^+ = \sum_{j=1}^{m} a_j q_j , \qquad D^- = \sum_{i=1}^{n} c_i p_i ,$$

where the c_i's are some positive integers satisfying the conditions

$$c_i \geq b_i + 1 ; \qquad \sum c_i = 3g - 2 + n + d .$$

Consider the complex vector space

$$L(-D) = \{\varphi \in H^0(M_g , \mathscr{M}^*) : (\varphi) \geq D\} \cup \{0\} .$$

Since the support of $D^+ = (dF)_0$ is nonempty, we note that nonzero constant functions may not lie in $L(-D)$. By the Riemann-Roch theorem

$$\begin{aligned} \dim L(-D) &= \deg(-D) - g + 1 + \dim L((dF) + D) \\ &= 1 + \dim L((dF) + D) \geq 1 . \end{aligned}$$

Thus there are nonconstant meromorphic functions belonging to $L(-D)$. Let G be such a function, and also let m' denote the number of distinct zeros of G. Using an argument totally similar to the one given in [Y1, pp. 708–710] we can find a complete conformal minimal immersion $M_g \backslash \Sigma \to \mathbb{R}^3$, where

$$\Sigma = \mathrm{support}(F)_\infty \cup \mathrm{support}(G)_0 .$$

But F has n distinct poles and G has m' distinct zeros. This establishes the equality in $(**)$. Since $(G)_0 \geq D^+ = \sum_{j=1}^{m} a_j q_j$, we must have $m' \geq m$, proving the first inequality in $(*)$. On the other hand, m' is at most equal to the degree of D^-: the degree of D is negative and G can have at most $\deg(D^-)$ many simple zeros. So

$$m' \leq \sum c_i = 3g - 2 + n + d .$$

The rest follows easily. □

Proof of Theorem. We first consider the case $g = 1$. Let M_1 be any complex torus, and also let $\wp(z)$ denote the Weierstrass function on it, where z is the (global) Euclidean coordinate on M_1. Let $p \in M_1$ be the lattice point so that

$$(\wp)_\infty = 2p .$$

Now $d\wp = \wp' \, dz$, and it is well known that $\wp'$ has three distinct simple zeros, say q_1, q_2, and q_3. Thus

$$(d\wp) = (q_1 + q_2 + q_3) - 3p .$$

Taking $F = \mathfrak{p}$ in the Lemma, we find that $d = 2$, $n = 1$, $m = 3$. Now take

$$D = (q_1 + q_2 + q_3) - 4p.$$

Then $G = \mathfrak{p}' \in L(-D)$ and $m' = 3$. It follows that $|\Sigma_{F,G}| = 4$, proving (a) of the Theorem. We now assume that $g > 1$. We will show that we can pick $F \in H^0(M_g, \mathscr{M}^*)$ so that

$$(\dagger) \qquad 2n_F + d_F + 3g - 2 \leq 4g.$$

Then the last inequality in $(*)$ will give (b) of the Theorem. Let $p \in M_g$ be any Weierstrass point. This means that the gap sequence at p is not $\{1, 2, \ldots, g\}$. In particular, there is a nongap $\tilde{d} \leq g$ at the point p, meaning that there is a meromorphic function $\widetilde{F}$ on M_g with $(\widetilde{F})_\infty = \tilde{d}p$. Take $F = \widetilde{F}$. Then $d = \tilde{d}$, $n = 1$, and $(\dagger)$ follows. Before proving (c) in general we first look at the case $g = 1$. We have the following

Observation. Let M_1 be any complex torus, and also let $\tilde{n} \geq 2$ be an integer. Then we can find a principal divisor on M_1 of the form

$$\sum p_i - \sum q_i, \qquad 1 \leq i \leq \tilde{n},$$

where the p_i's and the q_i's are distinct points of M_1.

The above observation is proved easily from Abel's theorem, which in this case states that a degree zero divisor on M_1 is principal if and only if the group sum (in the abelian group M_1) of the points of D is zero.

Given M_1 and any $n = \tilde{n} \geq 2$, we take F such that

$$(F) = \sum_{i=1}^{n} (p_i - q_i), \quad \text{the } p_i\text{'s and } q_i\text{'s are distinct}.$$

Then $(*)$ gives

$$(1) \qquad n + 1 \leq |\Sigma_F| \leq 3n + 1$$

since $d = n = \tilde{n}$ and $m \geq 1$. On the other hand, we can take F to be such that $(F)_\infty = dp$, where p is any point of M_1 and d is any integer ≥ 2. Then $n = 1$, and the Lemma gives

$$(2) \qquad 2 \leq |\Sigma_F| \leq d + 3.$$

The inequalities in (1), (2) prove (c) for the case $g = 1$. Now let M_g denote a Riemann surface of genus $g > 1$. Then, given any $d > g$, there is a meromorphic function F with $(F)_\infty = dp$, $p \in M_g$; just take p to be any non-Weierstrass point. Then $n = 1$ and the Lemma gives

$$(3) \qquad |\Sigma_F| \leq 3g + d.$$

On the other hand, given any $n \in \mathbb{Z}^+$, there is a meromorphic function F on M_g with n distinct poles (just add n "gap functions" at n distinct points). So given any $n \in \mathbb{Z}^+$ we can have

$$(4) \qquad n + 1 \leq |\Sigma_F|.$$

Combining (3), (4) we obtain the following

Proposition. *Given any Riemann surface M_g of genus g and any integer $d > g$, there exists a complete conformal minimal immersion $M_g \backslash \Sigma \to \mathbb{R}^3$ with*

$$d + 1 \leq |\Sigma| \leq 3g + d.$$

The above proposition easily implies (c) of the Theorem. □

3. Concluding remarks

One suspects that $\mathscr{P}(g) \supset \{N \in \mathbb{Z} : N \geq 4g\}$—a sort of deformation argument might work. On the other hand, it is unclear that $4g$ is actually the minimum of $\mathscr{P}(g)$. For example, one can reduce the degree of the meromorphic function used in producing the lower bound $4g$: on any M_g it is well known that there is a meromorphic function of degree $d \leq (g + 3)/2$. However, it seems difficult to determine the number of distinct poles of such a meromorphic function.

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References

[B] D. Bloss, *Elliptischefunktionen und vollstandige minimalflachen*, doctoral dissertation, Freien Universitat, Berlin, 1989.

[CO] S. Chern and R. Osserman, *Complete minimal surfaces in Euclidean n-space*, J. Analyse Math. **19** (1967), 15–34.

[GK] F. Gackstatter and R. Kunert, *Konstruktion vollstänger Minimalflächen von endlicher Gesamtkrümmung*, Arch. Rational Mech. Anal. **65** (1977), 289–297.

[JM] L. Jorge and W. Meeks, *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983), 203–221.

[KS] T. Klotz and L. Sario, *Existence of complete minimal surfaces of arbitrary connectivity and genus*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 42–44.

[M] X. Mo, *Gauss map and moduli space of minimal surfaces in Euclidean spaces*, Doctoral dissertation, Stanford University, Stanford, CA, 1990.

[O] R. Osserman, *A survey of minimal surfaces*, Van Nostrand, New York, 1969.

[Y1] K. Yang, *Meromorphic functions on a compact Riemann surface and associated complete minimal surfaces*, Proc. Amer. Math. Soc. **105** (1989), 706–711.

[Y2] K. Yang, *Complete and compact minimal surfaces*, Kluwer Academic Publishers, Boston, MA, 1989.

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