A CHARACTERIZATION OF HARMONIC ARAKELYAN SETS

M. GOLDSFNE AND W. H. OW

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Abstract. Given a closed subset $F$ of either $\mathbb{R}^N$, $N \geq 3$, or a Riemann surface, necessary and sufficient conditions are given so that every function continuous on $F$ and harmonic in the interior can be uniformly approximated on $F$ by globally defined harmonic functions.

1. Introduction

At a 1979 NATO conference in Durham, U.K., the following problem was posed by Goldstein [5]: Let $F$ be a closed subset of Euclidean space $\mathbb{R}^N$, $N \geq 2$. Call $F$ a set of harmonic approximation if every function continuous on $F$ and harmonic in the interior of $F$ can be uniformly approximated there by harmonic functions on $\mathbb{R}^N$. Give necessary and sufficient conditions so that $F$ is a set of harmonic approximation. In this direction necessary and sufficient conditions in $\mathbb{R}^N$ have been given by Shaginyan [11], but only if $F$ is nowhere dense. More recently, Bagby and Gauthier [3] have given other necessary and sufficient conditions on Riemann surfaces for arbitrary closed sets. Also, Shaginyan [12] has stated, without giving any proofs, results for tangential approximation on closed sets in $\mathbb{R}^N$. Still more recently, Shaginyan and Ladouceur [13] have stated necessary and sufficient conditions for arbitrary closed sets in $\mathbb{R}^N$, but their conditions are incorrect, as can be shown by an example.

In this paper we completely characterize sets $F$ of harmonic approximation in $\mathbb{R}^N$, $N \geq 3$, as well as on Riemann surfaces. Such sets $F$ are also called harmonic Arakelyan sets since they are harmonic analogues of closed sets in the plane for which the Arakelyan Theorem [1] for uniform approximation by entire functions applies. Before discussing our results we state some preliminaries.

Let $E$ be a nonempty subset of $\Omega$, where $\Omega$ denotes either $\mathbb{R}^N$, $N \geq 3$, or a Riemann surface. Let $E^* = E \cup \{\ast\}$ be the Alexandroff one-point compactification of $E$, where $\ast$ denotes the ideal point of $E$. We denote the interior, closure, and finite boundary of a set $E$ in $\Omega$ by $E^0$, $\overline{E}$, and $\partial E$, respectively; while $\widehat{E}$ denotes the union of $E$ and all of the relatively compact components of $\Omega - E$. The Euclidean norm of $x \in \mathbb{R}^N$ is denoted $\|x\|$.


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Let $C(E)$ be the set of all real-valued continuous functions on $E$, and $H(E)$ the set of real-valued functions harmonic on some open set containing $E$. Functions in $H(\Omega)$ are called entire harmonic functions. If $F$ is a closed subset of $\Omega$ we define $A(F) = C(F) \cap H(F^0)$, while $\overline{H}_F(\Omega)$ (respectively, $\overline{H}(F)$) will denote the uniform limits on $F$ of functions in $H(\Omega)$ (respectively, uniform limits of functions in $H(F)$).

The main goal of this paper is to prove

**Theorem 1.** Let $F$ be a closed subset of $\Omega$, where either $\Omega = \mathbb{R}^N$, $N \geq 3$, or $\Omega$ is a Riemann surface. Then $A(F) = \overline{H}_F(\Omega)$ if and only if the following conditions are satisfied:

1. (1.1) $\Omega - \hat{F}$ and $\Omega - (\hat{F})^0$ are thin at the same points.
2. (1.2) Every function $u \in A(F)$ can be extended to a function $\hat{u} \in A(\hat{F})$.
3. (1.3) $\Omega^* - \hat{F}$ is locally connected.

A proof of Theorem 1 is given in §2.

**Remark 1.** In the case of a Riemann surface Theorem 1 reduces to a theorem of Bagby and Gauthier [3] because Theorem 1(1.1) is easily seen to be satisfied on Riemann surfaces, since thinness is a local property, and thus [9, Theorem 10.14] applies. Hence, we shall confine ourselves to proving Theorem 1 in the case $N \geq 3$.

It should be noted that, compared to the Bagby-Gauthier Theorem [3] mentioned above, our theorem requires a thinness condition (1.1) analogous to that of Deny [6] for uniform harmonic approximation in a neighborhood of a compact set. Furthermore, since there is a substantial difference in the notion of thinness between dimension 2 and higher dimensions, the proof of Theorem 1 is quite different from that of the Bagby-Gauthier theorem.

**Remark 2.** Theorem 1 is a harmonic analogue of the following.

**Theorem (Arakelyan [1]).** Let $F$ be a closed set in the complex plane $\mathbb{C}$. Then the necessary and sufficient conditions so that every function $f(z)$ continuous on $F$ and analytic in $F^0$ is uniformly approximable on $F$ by entire functions are

(i) $\mathbb{C}^* - F$ is connected,
(ii) $\mathbb{C}^* - F$ is locally connected.

**Remark 3.** As noted earlier by Goldstein [5] Theorem 1 has applications to Problem 9.7 of Rubel [5] as well as other applications.

**Remark 4.** From Theorem 2 of Gauthier, Goldstein, and Ow [7] it follows that, for closed subsets $F$ of Riemann surfaces $\Omega$, the following is a necessary condition for $A(F) = \overline{H}_F(\Omega)$: For each bounded open set $V$ such that $\partial V \subset F$, either $V \subset F$ or $V \cap F = \emptyset$.

This condition is also necessary in dimension $N \geq 3$, but since it is not needed in the proof of Theorem 1 we shall omit its proof.

2. **Proof of Theorem 1**

2a. **Sufficiency.** Assume that conditions (1.1), (1.2), and (1.3) hold. Let $u \in A(F)$ and a constant $\epsilon > 0$ be given. By (1.2) we may extend $u$ to a function $\hat{u} \in A(\hat{F})$. By (1.1) it follows from [10, Theorem 3.10] that $A(\hat{F}) = \overline{H}(\hat{F})$. 

Hence, there exists a \( w \in H(\hat{F}) \) such that \( |\hat{u} - w| < \varepsilon/2 \) on \( \hat{F} \). Further, since \( (\mathbb{R}^N)^* - \hat{F} \) is connected and by (1.3) locally connected, there exists by a theorem of Gauthier, Goldstein, and Ow [8] a \( v \in H(\mathbb{R}^N) \) such that \( |w - v| < \varepsilon/2 \) on \( \hat{F} \). Hence, \( |\hat{u} - v| < \varepsilon \) on \( \hat{F} \) and this implies \( |u - v| < \varepsilon \) on \( F \). Thus, \( A(F) = \overline{H}_F(\mathbb{R}^N) \) as was to be shown.

2b. Necessity. Assume that \( A(F) = \overline{H}_F(\mathbb{R}^N) \). We first show that condition (1.1) is necessary. We first claim \( A(\hat{F}) = \overline{H}(\hat{F}) \). To see this, let \( u \in A(\hat{F}) \). Then \( u \) is continuous on \( F \). Further, if \( p \in F^0 \), then \( F \) and hence \( \hat{F} \) contain a neighborhood of \( p \); and so \( u \) is harmonic at \( p \) and thus \( u \in A(F) \). By assumption, \( u \) is the uniform limit on \( F \) of a sequence \( \langle h_n \rangle \) of entire harmonic functions. It remains to show that \( u \) is the uniform limit of \( \langle h_n \rangle \) on \( \hat{F} \). Let \( \varepsilon \) be a positive constant. Then there exists an integer \( J \) such that \( |u - h_k| < \varepsilon \) on \( F \) whenever \( k \geq J \). If \( E \) is any relatively compact component of \( \mathbb{R}^N - F \), then \( \partial E \subset F \) implies \( |u - h_k| < \varepsilon \) on \( \partial E \) and hence \( |h_m - h_k| < 2\varepsilon \) on \( \partial E \) if \( m, k \geq J \). By the maximum principle \( |h_m - h_k| < 2\varepsilon \) holds on \( \overline{E} \) and thus \( |h_m - h_k| < 2\varepsilon \) on \( \hat{F} \). Hence, there exists a function \( h \) on \( \hat{F} \) such that \( h_n \to h \) uniformly on \( \hat{F} \). So \( h \in A(\hat{F}) \), and since \( h \equiv u \) on the boundary of each relatively compact component of \( \mathbb{R}^N - F \), the maximum principle again implies \( u \equiv h \) on \( \hat{F} \) and hence \( A(\hat{F}) = H(\hat{F}) \). Finally, by [10, Theorem 3.10] \( A(\hat{F}) = H(\hat{F}) \) implies condition (1.1), as was to be shown.

Before showing the necessity of condition (1.2) we first prove

**Lemma 1.** If \( A(F) = \overline{H}_F(\mathbb{R}^N) \), \( N \geq 2 \), then necessarily \( \partial F = \partial \hat{F} \).

**Proof.** For a proof of the case \( N = 2 \) see [7].

We now prove the necessity of \( \partial F = \partial \hat{F} \) when \( N \geq 3 \). Observe that \( \partial \hat{F} \subset \partial F \) and that \( \partial F - \partial \hat{F} = \partial F \cap (\hat{F})^0 \). To obtain a contradiction we suppose that there exists a point \( p \in (\hat{F})^0 \cap \partial F \). Then there exists a neighborhood \( B_r(p) = \{ q : ||p - q|| < r \} \subset (\hat{F})^0 \). Within \( B_r(p) \) there are points \( q \in \mathbb{R}^N - F \) arbitrarily close to \( p \) since \( p \in \partial F \). Fix a \( q_0 \in (\mathbb{R}^N - F) \cap B_r(p) \). Now the function \( u(x) = ||x - q_0||^{2-N} \in A(F) \). Let \( \varepsilon \) be a positive constant satisfying

\[ 0 < \varepsilon < 1. \]

By hypothesis there exists an \( h \in H(\mathbb{R}^N) \) such that

\[ |h - u| < \varepsilon \quad \text{on} \quad F. \]

Since \( u \) is nonconstant on \( F \) we may assume \( h \) is also nonconstant on \( F \). We initially choose \( q_0 \in (\mathbb{R}^N - F) \cap B_r(p) \) so that

\[ ||p - q_0|| < \frac{\varepsilon}{2}. \]

Then \( ||x - q_0|| \geq \frac{\varepsilon}{2} \) for \( x \in \mathbb{R}^N - B_r(p) \) and so \( u(x) = ||x - q_0||^{2-N} \leq (\frac{\varepsilon}{2})^{2-N} \) for \( x \in \mathbb{R}^N - B_r(p) \). Hence, by (1) and (2),

\[ |h| \leq |h - u| + |u| \leq \varepsilon + (\frac{\varepsilon}{2})^{2-N} < 1 + (\frac{\varepsilon}{2})^{2-N}, \quad x \in F - B_r(p). \]

On the other hand, on \( F \cap B_r(p) \) we have by (1), (2) that

\[ |h| \geq |u| - |h - u| \geq |u| - \varepsilon \geq |u| - 1. \]
In addition to \( q_0 \) satisfying (3) we may further choose \( q_0 \) so close to \( p \) that
\[
\sup_{x \in F \cap B_r(p)} |u(x)| > 2\left[1 + \left(\frac{r}{2}\right)^{2-N}\right] + 1.
\]

By (5), (6),
\[
\sup_{x \in F \cap B_r(p)} |h(x)| \geq 2\left[1 + \left(\frac{r}{2}\right)^{2-N}\right].
\]
Finally, (4), (7) imply
\[
\sup_{x \in F} |h(x)| \text{ occurs in } F \cap B_r(p).
\]

Assume first that \( F \) is compact, and so \( \hat{F} \) is compact. By the maximum principle \( \sup_{F} |h| = \sup_{\hat{F}} |h| \). But by (8), \( \sup_{F} |h| \) occurs at a point of \( F \cap B_r(p) \) all of which are interior to \( \hat{F} \). Hence, \( \sup_{\hat{F}} |h| \) occurs at an interior point of \( \hat{F} \), contradicting \( h \) being nonconstant on \( F \).

Next assume \( F \) is unbounded. Choose an exhaustion of \( \mathbb{R}^N \) by balls \( T_j, T_{j+1} \subseteq T_j \), \( \mathbb{R}^N = \bigcup_{j} T_j \) so \( \partial T_j \cap F \neq \emptyset \) for all \( j \). Further, choose \( j \) large enough so that \( B_{r_0}(p) \subseteq T_j \) and \( u(x) = \|x - q_0\|^{2-N} < 1 \) for all \( x \in \mathbb{R}^N - T_j \). We apply the earlier compact case to \( F \cap T_j \) and again obtain a contradiction by violating the maximum principle. This completes the proof of the lemma.

We now show the necessity of condition (1.2). Assume \( A(F) = \overline{H}_F(R^N) \) and let \( u \in A(F) \). Now \( \partial F = \partial \hat{F} \) by Lemma 1 and hence \( u|_{\partial F} = u|_{\partial \hat{F}} \). Since \( A(F) = \overline{H}_F(R^N) \), we have by the maximum principle that \( u|_{\partial \hat{F}} \) extends to a function \( \hat{u} \in A(\hat{F}) \) and this \( \hat{u} \) is the desired extension of \( u \) to \( \hat{F} \). Note by Lemma 1 that no points of \( \partial F \cap (\hat{F})^0 \) are involved in the extension since \( \partial F \cap (\hat{F})^0 = \partial F - \partial \hat{F} = \emptyset \).

Finally, we show that condition (1.3) is necessary. We first need

Lemma 2 (Bagby-Gauthier [4]). Let \( W \) be a bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), and \( q \in W \). Denote by \( g_W(\cdot, q) \) the Green function for \( W \) with singularity at \( q \). Let \( u : \overline{W} - \{q\} \to \mathbb{R} \) be continuous, harmonic on \( W - \{q\} \), and suppose \( u - g_W(\cdot, q) \) has a removable singularity at \( q \). Then \( u(p) \geq 0 \) for all \( p \in \partial W \) implies \( u(p) - g_W(p, q) \geq 0 \) for all \( p \in W \).

We now show that (1.3) is necessary. Assume \( A(F) = \overline{H}_F(R^N) \) and, to obtain a contradiction, suppose \( (R^N)^* - \hat{F} \) is not locally connected. Then there exists a neighborhood \( U = \{p|\|p\| > R > 1\} \) of \( * \) and there exists points \( p \in (R^N)^* - \hat{F}, p \neq *, \) with \( \|p\| \) arbitrarily large and which cannot be joined to \( * \) by a curve in \( (R^N)^* - \hat{F} \) without hitting the ball \( B_R(0) = \{p|\|p\| < R\} \). Each of these points \( p \) thus belongs to a bounded component of \( [(R^N)^* - \hat{F}] \cap B_R(0) \) whose boundary consists of points of \( \hat{F} \) and the sphere \( S_R(0) = \{p|\|p\| = R\} \). We choose such a sequence \( \langle b_m \rangle \) such that
\[
m \leq \|b_m\|, \quad \|b_m\| < \frac{1}{3}\|b_{m+1}\|,
\]
and each \( b_m \) belongs to a bounded component \( D_m \) of \( [(R^N)^* - \hat{F}] \cap B_R(0) \), with \( 2\|b_m\| > \sup_{p \in D_m} \|p\| \). The conditions (9) imply that the components \( D_m \) are pairwise disjoint. Also note \( S_R(0) \cap \partial D_m \neq \emptyset \) for each \( m \) since
otherwise $\partial D_m \subset \hat{F}$, a contradiction, so we may choose a sequence $\langle a_m \rangle$ such that $a_m \in D_m$, $a_m \neq b_m$, and $\text{dist}(a_m, S_R(0)) < \frac{1}{m}$. We may assume $\langle a_m \rangle$ converges to some point $a \in S_R(0)$.

We follow the method of proof of Theorem 3.3 in [4] and let $g_m = g_{D_m}(\cdot, b_m)$, $m \geq 1$, be the Green function for $D_m$ with singularity at $b_m$. For each $m \geq 1$ choose a finite constant $c_m$ such that $c_m > m/g_m(a_m)$. By Lemma 2.6 in [2] there exists a function $u$ harmonic in $\mathbb{R}^N - \bigcup_{m=1}^{\infty} b_m$ such that for each $m$ the harmonic function $u - c_m g_m$ has a removable singularity at $b_m$. Note that $A(F) = H_F(\mathbb{R}^N)$ implies $A(F) = H_{\hat{F}}(\mathbb{R}^N)$ (cf. [4, Remark 4.1b]). Now $u \in A(\hat{F})$ and so by assumption there exists a function $v$ harmonic in $\mathbb{R}^N$ such that $0 \leq u - v \leq 1$ on $\hat{F}$. Let $c = \min_{S_{R(0)}} (u - v)$ and $d = \min(0, c)$. Recall $\partial D_m \subset \hat{F} \cup S_R(0)$. Note that $u - v - d \geq 0$ throughout $\partial D_m$. Since $u - v - d$ is continuous on $\bar{D}_m - \{b_m\}$ and harmonic on $D_m - \{b_m\}$, and $u - v - d - c_m g_m$ has a removable singularity at $b_m$, we conclude by Lemma 2 that $u - v - d - c_m g_m \geq 0$ throughout $D_m$. In particular, at $a_m$ we have $u(a_m) - v(a_m) \geq c_m g_m(a_m) + d \geq m + d$. Hence $u(a) - v(a) = \lim_{m \to \infty} u(a_m) - v(a_m) = \infty$, which is a contradiction. This completes the proof of the necessity in Theorem 1.

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We thank the referee for pointing out an error in our original manuscript in which the proof of the necessity of Theorem 1(1.3) was adapted from a dubious earlier argument for the necessity of (1.3) in [7]. We have corrected this, using an argument in a recent manuscript of Bagby and Gauthier [4].

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DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85287-0001

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824-0001