A CHARACTERIZATION OF HARMONIC ARAKELYAN SETS

M. GOLDSTEIN AND W. H. OW

(Communicated by J. Marshall Ash)

Abstract. Given a closed subset $F$ of either $\mathbb{R}^N$, $N \geq 3$, or a Riemann surface, necessary and sufficient conditions are given so that every function continuous on $F$ and harmonic in the interior can be uniformly approximated on $F$ by globally defined harmonic functions.

1. Introduction

At a 1979 NATO conference in Durham, U.K., the following problem was posed by Goldstein [5]: Let $F$ be a closed subset of Euclidean space $\mathbb{R}^N$, $N \geq 2$. Call $F$ a set of harmonic approximation if every function continuous on $F$ and harmonic in the interior of $F$ can be uniformly approximated there by harmonic functions on $\mathbb{R}^N$. Give necessary and sufficient conditions so that $F$ is a set of harmonic approximation. In this direction necessary and sufficient conditions in $\mathbb{R}^N$ have been given by Shaginyan [11], but only if $F$ is nowhere dense. More recently, Bagby and Gauthier [3] have given other necessary and sufficient conditions on Riemann surfaces for arbitrary closed sets. Also, Shaginyan [12] has stated, without giving any proofs, results for tangential approximation on closed sets in $\mathbb{R}^N$. Still more recently, Shaginyan and Ladouceur [13] have stated necessary and sufficient conditions for arbitrary closed sets in $\mathbb{R}^N$, but their conditions are incorrect, as can be shown by an example.

In this paper we completely characterize sets $F$ of harmonic approximation in $\mathbb{R}^N$, $N \geq 3$, as well as on Riemann surfaces. Such sets $F$ are also called harmonic Arakelyan sets since they are harmonic analogues of closed sets in the plane for which the Arakelyan Theorem [1] for uniform approximation by entire functions applies. Before discussing our results we state some preliminaries.

Let $E$ be a nonempty subset of $\Omega$, where $\Omega$ denotes either $\mathbb{R}^N$, $N \geq 3$, or a Riemann surface. Let $E^* = E \cup \{\ast\}$ be the Alexandroff one-point compactification of $E$, where $\ast$ denotes the ideal point of $E$. We denote the interior, closure, and finite boundary of a set $E$ in $\Omega$ by $E^0$, $\overline{E}$, and $\partial E$, respectively; while $\hat{E}$ denotes the union of $E$ and all of the relatively compact components of $\Omega - E$. The Euclidean norm of $x \in \mathbb{R}^N$ is denoted $\|x\|$. 


1991 Mathematics Subject Classification. Primary 31A05, 31B05.
Let $C(E)$ be the set of all real-valued continuous functions on $E$, and $H(E)$ the set of real-valued functions harmonic on some open set containing $E$. Functions in $H(\Omega)$ are called entire harmonic functions. If $F$ is a closed subset of $\Omega$ we define $A(F) \equiv C(F) \cap H(F^0)$, while $H_F(\Omega)$ (respectively, $H(F)$) will denote the uniform limits on $F$ of functions in $H(\Omega)$ (respectively, uniform limits of functions in $H(F)$).

The main goal of this paper is to prove

**Theorem 1.** Let $F$ be a closed subset of $\Omega$, where either $\Omega = \mathbb{R}^N$, $N \geq 3$, or $\Omega$ is a Riemann surface. Then $A(F) = H_F(\Omega)$ if and only if the following conditions are satisfied:

1. $\Omega - \hat{F}$ and $\Omega - (\hat{F})^0$ are thin at the same points.
2. Every function $u \in A(F)$ can be extended to a function $\hat{u} \in A(\hat{F})$.
3. $\Omega^* - \hat{F}$ is locally connected.

A proof of Theorem 1 is given in §2.

**Remark 1.** In the case of a Riemann surface Theorem 1 reduces to a theorem of Bagby and Gauthier [3] because Theorem 1(1.1) is easily seen to be satisfied on Riemann surfaces, since thinness is a local property, and thus [9, Theorem 10.14] applies. Hence, we shall confine ourselves to proving Theorem 1 in the case $N \geq 3$.

It should be noted that, compared to the Bagby-Gauthier Theorem [3] mentioned above, our theorem requires a thinness condition (1.1) analogous to that of Deny [6] for uniform harmonic approximation in a neighborhood of a compact set. Furthermore, since there is a substantial difference in the notion of thinness between dimension 2 and higher dimensions, the proof of Theorem 1 is quite different from that of the Bagby-Gauthier theorem.

**Remark 2.** Theorem 1 is a harmonic analogue of the following.

**Theorem (Arakelyan [1]).** Let $F$ be a closed set in the complex plane $\mathbb{C}$. Then the necessary and sufficient conditions so that every function $f(z)$ continuous on $F$ and analytic in $F^0$ is uniformly approximable on $F$ by entire functions are

1. $\mathbb{C}^* - F$ is connected,
2. $\mathbb{C}^* - F$ is locally connected.

**Remark 3.** As noted earlier by Goldstein [5] Theorem 1 has applications to Problem 9.7 of Rubel [5] as well as other applications.

**Remark 4.** From Theorem 2 of Gauthier, Goldstein, and Ow [7] it follows that, for closed subsets $F$ of Riemann surfaces $\Omega$, the following is a necessary condition for $A(F) = H_F(\Omega)$: For each bounded open set $V$ such that $\partial V \subset F$, either $V \subset F$ or $V \cap F = \emptyset$.

This condition is also necessary in dimension $N \geq 3$, but since it is not needed in the proof of Theorem 1 we shall omit its proof.

2. **Proof of Theorem 1**

2a. **Sufficiency.** Assume that conditions (1.1), (1.2), and (1.3) hold. Let $u \in A(F)$ and a constant $\varepsilon > 0$ be given. By (1.2) we may extend $u$ to a function $\hat{u} \in A(\hat{F})$. By (1.1) it follows from [10, Theorem 3.10] that $A(\hat{F}) = \overline{H(F)}$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Hence, there exists a \( w \in H(\hat{F}) \) such that \( |\hat{u} - w| < \varepsilon /2 \) on \( \hat{F} \). Further, since \((R^N)^* - \hat{F}\) is connected and by (1.3) locally connected, there exists by a theorem of Gauthier, Goldstein, and Ow [8] a \( v \in H(R^N) \) such that \( |w - v| < \varepsilon /2 \) on \( \hat{F} \). Hence, \( |\hat{u} - v| < \varepsilon \) on \( \hat{F} \) and this implies \( |u - v| < \varepsilon \) on \( F \). Thus, \( A(F) = \overline{H_F}(R^N) \) as was to be shown.

2b. Necessity. Assume that \( A(F) = \overline{H_F}(R^N) \). We first show that condition (1.1) is necessary. We first claim \( A(\hat{F}) = \overline{H}(\hat{F}) \). To see this, let \( u \in A(\hat{F}) \).

Then \( u \) is continuous on \( F \). Further, if \( p \in F^0 \), then \( F \) and hence \( \hat{F} \) contain a neighborhood of \( p \); and so \( u \) is harmonic at \( p \) and thus \( u \in A(F) \). By assumption, \( u \) is the uniform limit on \( F \) of a sequence \( \langle h_n \rangle \) of entire harmonic functions. It remains to show that \( u \) is the uniform limit of \( \langle h_n \rangle \) on \( \hat{F} \). Let \( \varepsilon \) be a positive constant. Then there exists an integer \( J \) such that \( |u-h_k| < \varepsilon \) on \( F \) whenever \( k \geq J \). If \( E \) is any relatively compact component of \( R^N - F \), then \( \partial E \subset F \) implies \( |u-h_k| < \varepsilon \) on \( \partial E \) and hence \( |h_m-h_k| < 2\varepsilon \) on \( \partial E \) if \( m \), \( k \geq J \). By the maximum principle \( |h_m-h_k| < 2\varepsilon \) holds on \( \overline{E} \) and thus \( |h_m-h_k| < 2\varepsilon \) on \( \hat{F} \). Hence, there exists a function \( h \) on \( \hat{F} \) such that \( h_n \to h \) uniformly on \( \hat{F} \). So \( h \in A(\hat{F}) \), and since \( h \equiv u \) on the boundary of each relatively compact component of \( R^N - F \), the maximum principle again implies \( u \equiv h \) on \( \hat{F} \) and hence \( A(\hat{F}) = H(\hat{F}) \). Finally, by [10, Theorem 3.10] \( A(\hat{F}) = H(\hat{F}) \) implies condition (1.1), as was to be shown.

Before showing the necessity of condition (1.2) we first prove

**Lemma 1.** If \( A(F) = \overline{H_F}(R^N) \), \( N \geq 2 \), then necessarily \( \partial F = \partial \hat{F} \).

**Proof.** For a proof of the case \( N = 2 \) see [7].

We now prove the necessity of \( \partial F = \partial \hat{F} \) when \( N \geq 3 \). Observe that \( \partial \hat{F} \subset \partial F \) and that \( \partial F - \partial \hat{F} = \partial F \cap (\hat{F})^0 \). To obtain a contradiction we suppose that there exists a point \( p \in (\hat{F})^0 \cap \partial F \). Then there exists a neighborhood \( B_r(p) = \{ q \ | \ |p-q| < r \} \subset (\hat{F})^0 \). Within \( B_r(p) \) there are points \( q \in R^N - F \) arbitrarily close to \( p \) since \( p \in \partial F \). Fix a \( q_0 \in (R^N - F) \cap B_r(p) \). Now the function \( u(x) = \|x-q_0\|^{-2-N} \in A(F) \). Let \( \varepsilon \) be a positive constant satisfying

\[
0 < \varepsilon < 1.
\]

By hypothesis there exists an \( h \in H(R^N) \) such that

\[
|h-u| < \varepsilon \quad \text{on } F.
\]

Since \( u \) is nonconstant on \( F \) we may assume \( h \) is also nonconstant on \( F \). We initially choose \( q_0 \in (R^N - F) \cap B_r(p) \) so that

\[
\|p-q_0\| < \frac{\varepsilon}{2}.
\]

Then \( \|x-q_0\| \geq \frac{\varepsilon}{2} \) for \( x \in R^N - B_r(p) \) and so \( u(x) = \|x-q_0\|^{-2-N} \leq (\frac{\varepsilon}{2})^{2-N} \) for \( x \in R^N - B_r(p) \). Hence, by (1) and (2),

\[
|h| \leq |h-u| + |u| \leq \varepsilon + (\frac{\varepsilon}{2})^{2-N} < 1 + (\frac{\varepsilon}{2})^{2-N}, \quad x \in F - B_r(p).
\]

On the other hand, on \( F \cap B_r(p) \) we have by (1), (2) that

\[
|h| \geq |u| - |h-u| \geq |u| - \varepsilon \geq |u| - 1.
\]
In addition to \(q_0\) satisfying (3) we may further choose \(q_0\) so close to \(p\) that
\[
\sup_{x \in F \cap B_r(p)} |u(x)| > 2[1 + (\frac{r}{2})^{2-N}] + 1.
\]

By (5), (6),
\[
\sup_{x \in F \cap B_r(p)} |h(x)| \geq 2[1 + (\frac{r}{2})^{2-N}] .
\]

Finally, (4), (7) imply
\[
\sup_{x \in F} |h(x)| \text{ occurs in } F \cap B_r(p).
\]

Assume first that \(F\) is compact, and so \(\hat{F}\) is compact. By the maximum principle \(\sup_{\hat{F}} |h| = \sup_{F} |h|\). But by (8), \(\sup_{\hat{F}} |h|\) occurs at a point of \(F \cap B_r(p)\) all of which are interior to \(\hat{F}\). Hence, \(\sup_{\hat{F}} |h|\) occurs at an interior point of \(\hat{F}\), contradicting \(h\) being nonconstant on \(F\).

Next assume \(F\) is unbounded. Choose an exhaustion of \(\mathbb{R}^N\) by balls \(T_j, \tilde{T}_j \subset T_{j+1}, \mathbb{R}^N = \bigcup_{j=1}^{\infty} T_j\) so \(\partial T_j \cap F \neq \emptyset\) for all \(j\). Further, choose \(j\) large enough so that \(\tilde{B}_r(p) \subset T_j\) and \(u(x) = \|x - q_0\|^{2-N} < 1\) for all \(x \in \mathbb{R}^N - T_j\). We apply the earlier compact case to \(F \cap T_j\) and again obtain a contradiction by violating the maximum principle. This completes the proof of the lemma.

We now show the necessity of condition (1.2). Assume \(A(F) = \overline{HF}(\mathbb{R}^N)\) and let \(u \in A(F)\). Now \(\partial F = \partial \hat{F}\) by Lemma 1 and hence \(u|_{\partial F} = u|_{\partial \hat{F}}\). Since \(A(F) = \overline{HF}(\mathbb{R}^N)\), we have by the maximum principle that \(u|_{\partial \hat{F}}\) extends to a function \(\hat{u} \in A(\hat{F})\) and this \(\hat{u}\) is the desired extension of \(u\) to \(\hat{F}\). Note by Lemma 1 that no points of \(\partial F \cap (\hat{F})^0\) are involved in the extension since \(\partial F \cap (\hat{F})^0 = \partial F - \partial \hat{F} = \emptyset\).

Finally, we show that condition (1.3) is necessary. We first need

**Lemma 2 (Bagby-Gauthier [4])**. Let \(W\) be a bounded domain in \(\mathbb{R}^N, N \geq 2\), and \(q \in W\). Denote by \(g_w(\cdot, q)\) the Green function for \(W\) with singularity at \(q\). Let \(u : W - \{q\} \rightarrow \mathbb{R}\) be continuous, harmonic on \(W - \{q\}\), and suppose \(u - g_w(\cdot, q)\) has a removable singularity at \(q\). Then \(u(p) \geq 0\) for all \(p \in \partial W\) implies \(u(p) - g_w(p, q) \geq 0\) for all \(p \in W\).

We now show that (1.3) is necessary. Assume \(A(F) = \overline{HF}(\mathbb{R}^N)\) and, to obtain a contradiction, suppose \((\mathbb{R}^N)^* - \hat{F}\) is not locally connected. Then there exists a neighborhood \(U = \{p|\|p\| > R > 1\}\) of \(*\) and there exists points \(p \in (\mathbb{R}^N)^* - \hat{F}, p \neq *,\) with \(\|p\|\) arbitrarily large and which cannot be joined to \(*\) by a curve in \((\mathbb{R}^N)^* - \hat{F}\) without hitting the ball \(B_R(0) = \{p|\|p\| < R\}\). Each of these points \(p\) thus belongs to a bounded component of \([(\mathbb{R}^N)^* - \hat{F}] \cap B_R(0)\) whose boundary consists of points of \(\hat{F}\) and the sphere \(S_R(0) = \{p|\|p\| = R\}\).

We choose such a sequence \(\langle b_m \rangle\) such that
\[
m \leq \|b_m\|, \quad \|b_m\| < \frac{1}{3}\|b_{m+1}\|,
\]
and each \(b_m\) belongs to a bounded component \(D_m\) of \([(\mathbb{R}^N)^* - \hat{F}] \cap B_R(0)\), with \(2\|b_m\| > \sup_{p \in D_m} \|p\|\). The conditions (9) imply that the components \(D_m\) are pairwise disjoint. Also note \(S_R(0) \cap \partial D_m \neq \emptyset\) for each \(m\) since
otherwise $\partial D_m \subset \bar{F}$, a contradiction, so we may choose a sequence $(a_m)$ such that $a_m \in D_m$, $a_m \neq b_m$, and $\text{dist}(a_m, S_R(0)) < \frac{1}{m}$. We may assume $(a_m)$ converges to some point $a \in S_R(0)$.

We follow the method of proof of Theorem 3.3 in [4] and let $g_m \equiv g_{D_m}(\cdot, b_m)$, $m \geq 1$, be the Green function for $D_m$ with singularity at $b_m$. For each $m \geq 1$ choose a finite constant $c_m$ such that $c_m > m/g_m(a_m)$. By Lemma 2.6 in [2] there exists a function $u$ harmonic in $R^N \setminus \bigcup_{m=1}^{\infty} b_m$ such that for each $m$ the harmonic function $u - c_m g_m$ has a removable singularity at $b_m$. Note that $A(F) = \mathcal{H}_F(R^N)$ implies $A(\hat{F}) = \mathcal{H}_{\hat{F}}(R^N)$ (cf. [4, Remark 4.1b]). Now $u \in A(\hat{F})$ and so by assumption there exists a function $v$ harmonic in $R^N$ such that $0 < u - v < 1$ on $F$. Let $c = \min_{S_R(0)}(u - v)$ and $d = \min(0, c)$. Recall $\partial D_m \subset \bar{F} \cup S_R(0)$. Note that $u - v - d \geq 0$ throughout $\partial D_m$. Since $u - v - d$ is continuous on $\bar{D}_m \setminus \{b_m\}$ and harmonic on $D_m \setminus \{b_m\}$, and $u - v - d - c_m g_m$ has a removable singularity at $b_m$, we conclude by Lemma 2 that $u - v - d - c_m g_m \geq 0$ throughout $D_m$. In particular, at $a_m$ we have $u(a_m) - v(a_m) \geq c_m g_m(a_m) + d \geq m + d$. Hence $u(a) - v(a) = \lim_{m \to \infty} u(a_m) - v(a_m) = \infty$, which is a contradiction. This completes the proof of the necessity in Theorem 1.

ACKNOWLEDGMENT

We thank the referee for pointing out an error in our original manuscript in which the proof of the necessity of Theorem 1(1.3) was adapted from a dubious earlier argument for the necessity of (1.3) in [7]. We have corrected this, using an argument in a recent manuscript of Bagby and Gauthier [4].

REFERENCES


DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85287-0001

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824-0001