AN EXTENSION OF NORM INEQUALITIES
FOR INTEGRAL OPERATORS ON CONES WHEN $0 < p < 1$

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Abstract. We extend our recent results concerning norm inequalities on cones to include the case when $0 < p < 1$.

In this note, we let $V$ be a homogeneous cone in $\mathbb{R}^n$. $V$ defines a partial ordering in $\mathbb{R}^n$ in such a way that $x <_V y$ if and only if $y - x \in V$. The cone interval $(a, b)$ is thus given by $(a, b) = \{x \in V : a <_V x <_V b\}$. For $x \in V$ we define $\Delta_V(x) = \int_{(0, x)} dy$.

Let $G(V)$ denote the automorphism group of $V$ and $f : V \to \mathbb{R}^+$ be a $V$-homogeneous function of order $\beta$. It is known (see [2, 5]) that if $f(x)$ is not identically 0 then $f(x) = c(\Delta_V(x))^\beta$ for all $x \in V$.

A $*$-function on $V$ is a mapping $x \mapsto x^*$ such that $x^* = -\text{grad} \log \phi_V(x)$, where $\phi_V(x)$ is the characteristic function of $V$. We have (see [1, 4]) that $(x^*)^* = x$ and the Jacobian determinant $|\partial_x x^*| = c\Delta_V^{-2}(x)$, where $c$ is a constant depending on $V$.

Let $V^*$ be the dual of $V$ and $G(V \to V^*)$ be the group of linear transforms mapping $V$ onto $V^*$. A homogeneous cone $V$ is called a domain of positivity if there is an $S \in G(V \to V^*)$ so that $S$ is symmetric and positive definite. It can be shown (see [4, 5]) that if $V$ is a domain of positivity then $x <_V y \Leftrightarrow y^* <_V x^*$.

We shall continue to consider integral operators of the form

$$Kf(x) = \int_V k(x, y)f(y)\, dy, \quad x \in V,$$

and

$$K^*f(y) = \int_V k(x, y)f(x)\, dx, \quad y \in V,$$

where $f : V \to \mathbb{R}^+$ and $k : V \times V \to \mathbb{R}^+$ is $(V \times V)$-homogeneous of order $\beta$; that is,

$$k(Ax, Ay) = |A|^\beta k(x, y) \quad \forall A \in G(V).$$

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We have shown (see [5]) the following general theorem concerning the $L^p$ boundedness of an integral operator on a cone:

**Theorem 1.** Let $V$ be a homogeneous cone in $\mathbb{R}^n$ and $1 \leq p \leq q < \infty$. Assume that the kernel $k(x, y) : V \times V \to \mathbb{R}^+$ is $(V \times V)$-homogeneous of order $\beta$. If, for some $\alpha, \gamma \in \mathbb{R}$, $K \Delta_V^\alpha(x) < \infty$ and

$$
\int_V k_{q/p}(x, y) \Delta_V^{q-p+((\alpha+\beta+1)q/p')}(x) \, dx < \infty
$$

for $x, y \in V$, where $1/p + 1/p' = 1$, then

$$
\left( \int_V \Delta_V^{q-p}(x)(Kf(x))^q \, dx \right)^{1/q} \leq c \left( \int_V f^p(x) \Delta_V^{\beta p+(\gamma+1)p-1}(x) \, dx \right)^{1/p}.
$$

In this note, we extend the preceding result to the case $0 < p < 1$.

**Theorem 2.** Let $V \subset \mathbb{R}^n$ be a homogeneous cone and $0 < p < 1$. Assume that $k(x, y) : V \times V \to \mathbb{R}^+$ is $(V \times V)$-homogeneous of order $-1$. If there exist $\alpha, \gamma \in \mathbb{R}$ such that

1. $K \Delta_V^\alpha(x) < \infty$
2. $K^* \Delta_V^{-p+\alpha/p'}(y) < \infty$

for $x, y \in V$, then

3. $\int_V \Delta_V^{-p}(x)(Kf(x))^p \, dx \geq c \int_V f^p(x) \Delta_V^{-p}(x) \, dx$,

in the sense that if the integral on the left is finite, then the integral on the right is also finite and the inequality holds.

**Proof.** We show first that, for some constant $c > 0$,

$$
(Kf(x))^p \geq c \cdot K(f^p \cdot \Delta_V^{-\alpha p/p'})(x) \cdot \Delta_V^{\alpha p/p'}(x).
$$

In fact, since $(1/p) > 1$, we use Hölder’s inequality to obtain that

$$
\int_V k(x, y)f^p(y)\Delta_V^{-\alpha p/p'}(y) \, dy
\begin{align*}
&= \int_V (k(x, y)f(y))^p \cdot k^{-p/p'}(x, y)\Delta_V^{-\alpha p/p'}(y) \, dy \\
&\leq \left( \int_V (k(x, y)f(y)) \, dy \right)^p \cdot \left( \int_V (k^{-p/p'}(x, y)\Delta_V^{-\alpha p/p'}(y))^{1/(1-p)} \, dy \right)^{1-p} \\
&= (Kf(x))^p \cdot (K \Delta_V^\alpha(x))^{1-p}.
\end{align*}
$$

Note that, because of assumption (1), $K \Delta_V^\alpha(x)$ is $V$-homogeneous of order $\alpha$, and so we have $K \Delta_V^\alpha(x) = c \cdot \Delta_V^\alpha(x)$, for some constant $c$. Therefore,

$$
\int_V k(x, y)f^p(y)\Delta_V^{-\alpha p/p'}(y) \, dy \leq c \cdot (Kf(x))^p \cdot \Delta_V^{\alpha(1-p)}(x),
$$

and then we have (4).
Using (4), we have that

\[
\int_V \Delta_V^{-p}(x) (Kf(x))^p \, dx \\
\geq c \cdot \int_V \Delta_V^{-p}(x) (\Delta_V^\alpha(x))^{p/p'} \left( \int_V k(x, y) f^p(y) \Delta_V^{-\alpha/p'}(y) \, dy \right) \, dx \\
= c \cdot \int_V f^p(y) \Delta_V^{-\alpha/p'}(y) \left( \int_V \Delta_V^{-p+(\alpha/p')}(x) k(x, y) \, dx \right) \, dy.
\]

Note that, because of (2), \( K^* \Delta_V^{-p+(\alpha/p')}(y) \) is \( V \)-homogeneous of degree \( \gamma - p + (\alpha/p') \), and hence we have \( K^* \Delta_V^{-p+(\alpha/p')}(y) = c\Delta_V^{-p+(\alpha/p')}(y) \), for some constant \( c \). Therefore,

\[
\int_V f^p(y) \Delta_V^{-\alpha/p'}(y) \left( \int_V \Delta_V^{-p+(\alpha/p')}(x) k(x, y) \, dx \right) \, dy \\
= c \int_V f^p(y) \Delta_V^{-p}(y) \, dy,
\]

and thus (3) holds.

Let \( \Sigma = \{ x \in V : |x| = 1 \} \). Define \( \sigma_0(V) = \inf\{ \alpha \in \mathbb{R} : \int_\Sigma \Delta_V^\alpha(t) \, dt < \infty \} \) and \( \sigma(V) = \max\{-1, \sigma_0\} \). It is known (see [2]) that if \( \alpha > \sigma(V) \), then \( \int_{(0, x)} \Delta_V^\alpha(y) \, dy < \infty \) for \( x \in V \).

We have the following generalization of Hardy's inequality in the case \( 0 < p < 1 \).

**Theorem 3.** Let \( V \) be a domain of positivity in \( \mathbb{R}^n \) and \( 0 < p < 1 \). If \( \gamma > (1 + \sigma(V^*))((p - 1) + \sigma(V) + p) \), then

\[
\int_V \Delta_V^{-p}(x) \left( \int_{(x, \infty)} f(y) \Delta_V^{-1}(y) \, dy \right)^p \, dx \geq c \int_V f^p(x) \Delta_V^{-p}(x) \, dx.
\]

**Proof.** Let \( k(x, y) = \Delta_V^{-1}(y) \chi_{(x, \infty)}(y) \) for \( x, y \in V \). Clearly, \( k(x, y) \) is \( (V \times V') \)-homogeneous of order \(-1\). Let \( \gamma \) be given so that \( \gamma > (1 + \sigma(V^*)) \times (p - 1) + \sigma(V) + p \). It follows that \( (p'/p) (\gamma - \sigma(V) - p) < 1 + \sigma(V^*) \). Let \( \alpha \) be a number so that \( \alpha < -1 - \sigma(V^*) \).

Since \( V \) is a domain of positivity, \( x <_V y \Leftrightarrow y^* <_V x^* \). Note that \( \Delta_V(y) = c \cdot \Delta_V^{-1}(y^*) \) and \( \partial_{y^*} y = c \cdot \Delta_V^{-2}(y^*) \). So if we introduce a change of variable \( z = y^* \), then we have that

\[
K \Delta_V^\alpha(x) = \int_V \Delta_V^{-1}(y) \chi_{(x, \infty)}(y) \Delta_V^\alpha(y) \, dy = \int_{(x, \infty)} \Delta_V^{-1+\alpha}(y) \, dy = c \int_{(0, x^*)} \Delta_V^{-1-\alpha}(z) \, dz.
\]

By the choice of \( \alpha \) the last integral is finite for any \( x \in V \).
We also have that
\[ K^* \Delta_{V}^{-p+(\alpha p/p')} (y) = \int_{V} \Delta_{V}^{-1} (y) \Delta_{V}^{-p+(\alpha p/p')} (x) \chi_{(x, \infty)} (y) \, dx \]
\[ = \Delta_{V}^{-1} (y) \int_{(0, y)} \Delta_{V}^{-p+(\alpha p/p')} (x) \, dx. \]

Note that since \(-p'/p)(\gamma - \sigma(V) - p) > \alpha \) and \( p' < 0 \), we have \( \gamma > \sigma(V) + p - (\alpha p/p') \). Thus the last integral above is finite for any \( y \in V \).

By Theorem 2, we conclude that
\[ \int_{V} \Delta_{V}^{-p}(x) \left( \int_{(x, \infty)} f(y) \Delta_{V}^{-1}(y) \, dy \right)^{p} \, dx \geq c \int_{V} f^{p}(x) \Delta_{V}^{-p}(x) \, dx. \]

**Theorem 4.** Let \( V \) be a domain of positivity in \( \mathbb{R}^{n} \) and \( 0 < p < 1 \). If \( \gamma < (1 - \sigma(V))(p - 1) - \sigma(V^*) \), then
\[ \int_{V} \Delta_{V}^{-p}(x) \left( \frac{1}{\Delta_{V}(x)} \int_{(0, x)} f(y) \, dy \right)^{p} \, dx \geq c \int_{V} f^{p}(x) \Delta_{V}^{-p}(x) \, dx. \]

**Proof.** Let \( k(x, y) = \Delta_{V}^{-1}(x) \chi_{(0, x)} (y) \) for \( x, y \in V \). Clearly, \( k(x, y) \) is \((V \times V)-\text{homogeneous of order } -1\). Let \( \gamma \) be given so that \( \gamma < (1 - \sigma(V)) \times (p - 1) - \sigma(V^*) \). It follows that \( (p'/p)(-\gamma + p - \sigma(V^*) - 1) < \sigma(V) \). Let \( \alpha \) be a number so that \( \alpha > \sigma(V) \). Then it follows that \(-\gamma + p - (\alpha p/p') - 1 > \sigma(V^*) \).

Since \( V \) is a domain of positivity, \( x \prec_{V} y \iff y^* \prec_{V^*} x^* \). Note also that \( \Delta_{V}(x) = c \cdot \Delta_{V^*}^{-1}(x^*) \) and \( \partial_{x^*} x = c \cdot \Delta_{V^*}^{-2}(x^*) \). So if we introduce a change of variable \( z = x^* \), we then have that
\[ K^* \Delta_{V}^{-p+(\alpha p/p')} (y) = \int_{V} \Delta_{V}^{-1}(x) \chi_{(0, x)} (y) \Delta_{V}^{-p+(\alpha p/p')} (x) \, dx \]
\[ = \int_{(y, \infty)} \Delta_{V}^{-p+(\alpha p/p')-1}(x) \, dx = c \int_{(0, y^*)} \Delta_{V^*}^{-p-(\alpha p/p')-1}(z) \, dz. \]

Because \(-\gamma + p - (\alpha p/p') - 1 > \sigma(V^*)\), the last integral is finite for any \( y \in V \).

We also have that
\[ K \Delta_{V}^{p}(x) = \Delta_{V}^{-1}(x) \int_{(0, x)} \Delta_{V}^{p}(y) \, dy. \]

Because \( \alpha > \sigma(V) \), the integral above is finite for any \( x \in V \).

By Theorem 2, we conclude that
\[ \int_{V} \Delta_{V}^{-p}(x) \left( \frac{1}{\Delta_{V}(x)} \int_{(0, x)} f(y) \, dy \right)^{p} \, dx \geq c \int_{V} f^{p}(x) \Delta_{V}^{-p}(x) \, dx. \]

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