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ON CLOSED SUBSPACES OF ω^*

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ABSTRACT. It is consistent with the negation of the Continuum Hypothesis that closed subsets of the space ω^* are exactly compact 0-dimensional F-spaces of weight $\leq c$.

1

A Tychonoff space X is called an F-space if each cozero subset of X is C^*-embedded (see [GJ]). If X is compact and 0-dimensional, then X is an F-space if and only if the Boolean algebra of open-closed subsets of X has no countable gaps. The remainder $\omega^* = \beta[\omega]\backslash\omega$ and all of its closed subsets are compact 0-dimensional F-spaces. Assuming CH (the continuum hypothesis) Louveau proved in [L] that conversely

(F) Each compact 0-dimensional F-space of weight $\leq c$ can be embedded into ω^* (and hence into $\beta[\omega]$).

On the other hand, assuming Martin's Axiom plus $c = \omega_2$, van Douwen and van Mill [vDvM] produced a counterexample for (F) and conjectured that (F) is in fact equivalent to CH.

In this note we prove the following

Theorem. *The statement* (F) *holds in a model of set theory obtained from a model V of CH by adding ω_2 Cohen reals.*

Thus, (F) is consistent with $c = \omega_2$ and, hence, not equivalent to CH. This answers negatively Question 17 in [vMR].

In the proof we replace (F) by the following equivalent formulation:

> Each Boolean algebra of cardinality c having no countable gaps is a homomorphic image of the algebra $P(\omega)/fin$ (and hence of $P(\omega)$).

2

Let V be a model of set theory satisfying CH and $V_1 = V[G]$ a model obtained from V by adding ω_1 Cohen reals. Of course, we have

$$\mathbb{A}^V \subseteq \mathbb{A}^{V_1}, \quad \text{where } \mathbb{A} = P(\omega)/fin.$$

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Lemma. *Let $\mathbb{B} \in V_1$ be an algebra of cardinality c having no countable gaps. Each homomorphism $h \in V$ from $\mathbb{A}^V$ onto a subalgebra $\mathbb{B}_0 \subseteq \mathbb{B}$ can be extended to a homomorphism in V_1 from $\mathbb{A}^{V_1}$ onto $\mathbb{B}$.*

Proof. We may assume that h is defined on $P(\omega) \cap V$ and $h(s) = 0$ for $s \in fin$. Consider a subalgebra of the form

$$\mathscr{A} = \mathbb{A}^V[a_\alpha : \alpha < \gamma],$$

where $\gamma < \omega_1$, i.e., $\mathscr{A}$ is generated by $\mathbb{A}^V$ and countably many new sets a_α. Each new set a determines a (filled) gap

$$L_a = \langle \{u \in \mathscr{A} : u \leq a\}; \ \{v \in \mathscr{A} : a \leq v\}\rangle,$$

which is countably generated. Indeed, each element $u \in \mathscr{A}$ is a finite sum of elements of the form $x \cap a(\varepsilon)$, where $x \in V$ and

$$a(\varepsilon) = \varepsilon(\alpha_1) \cdot a_{\alpha_1} \cap \cdots \cap \varepsilon(\alpha_n) \cdot a_{\alpha_n}$$

for an $\varepsilon : \{\alpha_1, \dots, \alpha_n\} \to \{-1, +1\}$. Let $\mathbb{C}$ be a countable subforcing such that $\mathscr{A}$, $a \in V[G \cap \mathbb{C}]$. For an $X \in V^{(\mathbb{C})}$ and $p \in \mathbb{C}$ let

$$X_p = \{i \in \omega : p \Vdash \text{“}i \in X\text{”}\}.$$

Thus, if $x \in V$ and $p \Vdash$ “$x \subseteq X$”, then $x \subseteq X_p$.

Now if $x \cap a(\varepsilon) \subseteq_* a$, i.e.,

$$p \Vdash \text{“}x \cap a(\varepsilon) \subseteq a \cup k\text{”}$$

for some $p \in G \cap \mathbb{C}$ and $k \in \omega$, then we have

$$x \cap a(\varepsilon) \subseteq X(p, \varepsilon, k) \cap a(\varepsilon) \subseteq a \cup k,$$

where $X(p, \varepsilon, k) = [(a \cup k) \cup (\omega \backslash a(\varepsilon))]_p$ are in V. Hence, the lower part of L_a is generated by countably many elements (finite unions of the sets $X(p, \varepsilon, k) \cap a(\varepsilon)$). Obviously, the same holds for the upper part of L_a.

Now, we extend h as follows. Since CH holds in V_1 we have $\mathbb{A}^{V_1} = \bigcup_{\alpha<\omega_1} \mathbb{A}_\alpha$, where

$$\begin{aligned} \mathbb{A}_0 &= \mathbb{A}^V, \\ \mathbb{A}_{\alpha+1} &= \mathbb{A}_\alpha[a_\alpha] \quad \text{for some } a_\alpha \in \mathbb{A}^{V_1}, \\ \mathbb{A}_\alpha &= \bigcup_{\beta<\alpha} \mathbb{A}_\beta \quad \text{for limit } \alpha < \omega_1. \end{aligned}$$

Similarly, there are generators $b_\alpha \in \mathbb{B}$ so that $\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbb{B}_\alpha$, with analogous properties.

Assume inductively that h has been already extended to an $h_\alpha : \mathscr{A}_\alpha \to \mathscr{B}_\alpha$ (onto), where both $\mathscr{A}_\alpha$, $\mathscr{B}_\alpha$ are countably generated over V and $\mathbb{A}_\alpha \subseteq \mathscr{A}_\alpha$, $\mathbb{B}_\alpha \subseteq \mathscr{B}_\alpha$. Let a be the earliest among the a_β, which is not in $\mathscr{A}_\alpha$. As stated above the gap L_a is countably generated in $\mathscr{A}_\alpha$, say by u_n (the lower part) and v_m (the upper part). Since $\mathbb{B}$ has no countable gaps, there is a $b \in \mathbb{B}$ such that

$$h_\alpha(u_n) \leq b \leq h_\alpha(v_m) \quad \text{for all } n, m \in \omega.$$

Hence for all $u, v \in \mathscr{A}_\alpha$ we have

$$u \leq a \quad \text{implies} \quad h_\alpha(u) \leq b$$

and

$$v \geq a \quad \text{implies} \quad h_\alpha(v) \geq b .$$

Now applying a theorem on extension of homomorphisms (see [S, 12.2, p. 36]) we infer that h_α has an extension $h_{\alpha+1}\colon \mathscr{A}_\alpha[a] \to \mathscr{B}_\alpha[b]$. If $b_\alpha \notin \mathscr{B}_\alpha[b]$ we extend $h_{\alpha+1}$ further as follows. There is a countable subforcing $\mathbb{C}$ such that $\mathscr{A}_\alpha[a] \in V[G \cap \mathbb{C}]$, and hence there is a set $A \in V_1$, which is Cohen generic over $V[G \cap \mathbb{C}]$. Now $u \leq A \leq v$ holds for finite u and cofinite v only, and hence (again using the Sikorski theorem) $h_{\alpha+1}$ can be extended to $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha[a, A]$ with $h_{\alpha+1}(A) = b_\alpha$. Thus, we have subalgebras $\mathscr{A}_{\alpha+1} \supseteq \mathbb{A}_\alpha$ and $\mathscr{B}_{\alpha+1} \supseteq \mathbb{B}_\alpha$ countably generated over V and a homomorphism from $\mathscr{A}_{\alpha+1}$ onto $\mathscr{B}_{\alpha+1}$ extending h_α.

The case of a limit stage α is obvious and the proof is finished.

3

Let $\kappa \in V$ be a regular uncountable cardinal and $\mathbb{P}_\kappa = \sum_{\alpha<\kappa} \mathbb{P}_\alpha$ a finite support c.c.c.-iteration. Let us note the following

Remark. If $f \in V^{(\mathbb{P}_\kappa)}[H]$ is a finitary operation $f\colon \kappa^m \to \kappa$, then there is a c.u.b. $N_f \subseteq \kappa$ such that for each $\gamma \in N_f$, $f \restriction \gamma \in V[H \cap \mathbb{P}_\gamma]$ and γ is closed under f.

Indeed, we choose a name $\underline{f} \in V^{(\mathbb{P}_\kappa)}$ of the form

$$\underline{f} = \{\langle \langle \overline{\alpha}, F(\overline{\alpha}, q)\rangle^{(\mathbb{P}_\kappa)}, q\rangle \colon \overline{\alpha} < \kappa \text{ and } q \in A(\overline{\alpha})\}$$

where $\overline{\alpha} = \langle \alpha_1, \dots, \alpha_m \rangle$, $A(\overline{\alpha}) \subseteq \mathbb{P}_\kappa$ is an antichain, and

$$F\colon \bigcup_{\overline{\alpha}<\kappa} \{\overline{\alpha}\} \times A(\overline{\alpha}) \to \kappa$$

is a function such that

$$q \Vdash \text{“}\underline{f}(\overline{\alpha}) = F(\overline{\alpha}, q)\text{”}$$

for each $q \in A(\overline{\alpha})$.

If we define

$$N_f = \{\gamma < \kappa \colon \forall \overline{\alpha} < \gamma \forall q \in A(\overline{\alpha})[A(\overline{\alpha}) \subseteq \mathbb{P}_\gamma \text{ and } F(\overline{\alpha}, q) < \gamma]\},$$

then N_f is a c.u.b. in κ, and if

$$\underline{f} \restriction \gamma = \{\langle \langle \overline{\alpha}, F(\overline{\alpha}, q)\rangle, q\rangle \in \underline{f} \colon \overline{\alpha} < \gamma \text{ and } q \in A(\overline{\alpha})\}$$

then $\underline{f} \restriction \gamma \in V^{(\mathbb{P}_\gamma)}$ and

$$(\underline{f} \restriction \gamma)[H \cap \mathbb{P}_\gamma] = f \restriction \gamma^m \quad \text{for each } \gamma \in N_f .$$

Hence, N_f is as required.

Let V_γ denote the model obtained from V (CH holds in V) by adding γ Cohen reals.

Corollary. *If $\mathbb{B} \in V_{\omega_2}$ is a Boolean algebra of cardinality $c = \omega_2$ having no countable gaps, then $\mathbb{B} = \bigcup_{\gamma \in N} \mathbb{B}_\gamma$, where*

(1) *N is unbounded in ω_2 and closed under ω_1-limits;*
(2) *$\mathbb{B}_\gamma \in V_\gamma$ is of cardinality $c = \omega_1$ and has no countable gaps in V_γ;*
(3) *the sequence $\{\mathbb{B}_\gamma \colon \gamma \in N\}$ is increasing and $\mathbb{B}_\gamma = \bigcup_{\alpha<\gamma} \mathbb{B}_\alpha$ for each ω_1-limit $\gamma \in N$.*

Proof. By the remark we have a decomposition

$$\mathbb{B} = \bigcup_{\gamma \in N_0} \mathbb{B}_\gamma , \qquad \mathbb{B}_\gamma \in V_\gamma ,$$

so (1) and (3) are satisfied. Since $\mathbb{B}$ has no countable gaps, for each $\gamma \in N_0$ there is a $\delta \in N_0$, such that each countable gap $L \subseteq \mathbb{B}_\gamma$, $L \in V_\gamma$, is filled in $\mathbb{B}_\delta$. Repeating this ω_1 times we see that the set

$$N = \{\gamma \in N_0 \colon \mathbb{B}_\gamma \text{ has no countable gaps in } V_\gamma\}$$

is unbounded and, clearly, closed under ω_1-limits.

Now we can finish the proof of our Theorem.

For a given $\mathbb{B} \in V_{\omega_2}$ let N be as described above and suppose that we have a homomorphism

$$h_\gamma \colon \mathbb{A} \cap V_\gamma \to \mathbb{B}_\gamma \quad \text{onto } \mathbb{B}_\gamma$$

in V_γ (if $\gamma = \inf N$, there exists such, by the Louveau theorem). If δ is the successor of γ in N we apply the lemma of §2 and obtain an extension h_δ of h_γ mapping $\mathbb{A} \cap V_\delta$ onto $\mathbb{B}_\delta$ (we may assume that $\beta \backslash \alpha$ is uncountable for all $\alpha, \beta \in N$, $\alpha < \beta$).

If γ is an ω_1-limit

$$\gamma = \sup_{\alpha < \omega_1} \gamma_\alpha , \quad \text{where } \{\gamma_\alpha \colon \alpha < \omega_1\} \subseteq N ,$$

then

$$\mathbb{A} \cap V_\gamma = \bigcup_{\alpha < \omega_1} \mathbb{A} \cap V_{\gamma_\alpha} , \qquad \mathbb{B}_\gamma = \bigcup_{\alpha < \omega_1} \mathbb{B}_{\gamma_\alpha}$$

and hence we take $h_\gamma = \bigcup_{\alpha < \omega_1} h_{\gamma_\alpha}$. Finally, let $\{\gamma_n \colon n < \omega\} \subseteq N$ be an increasing sequence and $\gamma = \sup_{n<\omega} \gamma_n$, in N. Then $h = \bigcup_{n<\omega} h_{\gamma_n}$ maps $A = \bigcup_{n<\omega} \mathbb{A} \cap V_{\gamma_n}$ onto $\bigcup_{n<\omega} \mathbb{B}_{\gamma_n} \subseteq \mathbb{B}_\gamma$. Note that each countably generated subalgebra

$$A[a_\alpha \colon \alpha < \beta] \subseteq \mathbb{Q} \cap V_\gamma$$

is the union of $\mathbb{A} \cap V_{\gamma_n}[a_\alpha \colon \alpha < \beta]$ and hence each gap L_a is countably generated in A over V_α, where $\alpha = \sup_{n<\omega} \gamma_n$ (since it is countably generated in each $\mathbb{A} \cap V_{\gamma_n}$). Thus, the lemma of §2 applies also in this case and h can be extended to a homomorphism

$$h_\gamma \colon \mathbb{A} \cap V_\gamma \to \mathbb{B}_\gamma \quad \text{onto } \mathbb{B}_\gamma ,$$

which completes the proof.

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