MEASURABILITY PROPERTIES OF SETS OF VITALI'S TYPE

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Abstract. Assume a group $G$ acts on a set. Given a subgroup $H$ of $G$, by an $H$-selector we mean a selector of the set of all orbits of $H$. We investigate measurability properties of $H$-selectors with respect to $G$-invariant measures.

Let us fix a set $X$ and a group $G$ acting on it. By $\mu$ we denote a $G$-invariant countably additive measure on $X$. The most common example of such a situation is an invariant measure on a group acting on itself by translations. Let $H$ be a subgroup of $G$. By an $H$-selector (sometimes called a set of Vitali's type for $H$) we understand a set having exactly one point in common with each orbit of $H$. Measurability properties of selectors were first systematically studied by Cichoń, Kharazishvili, and Weglorz in [1].

Selectors are extremely useful in constructing sets nonmeasurable with respect to an invariant measure. The first example of a Lebesgue nonmeasurable set, due to Vitali [8], is just a $Q$-selector where $Q$ is the group of rationals. Also for any finite invariant diffused measure on a group (acting on itself by translations) any $H$-selector for a countable subgroup $H$ is nonmeasurable. In fact, in both cases above the constructed sets are nonmeasurable with respect to any invariant extension of a given measure. Kharazishvili in [3] and Erdős and Mauldin in [2] constructed a nonmeasurable set for any $\sigma$-finite invariant measure. Their example is the union of a family of $H$-selectors where $H$ is a subgroup of cardinality $\omega_1$. Strengthening the result from [2, 3] the author constructed in [6] sets nonmeasurable with respect to any invariant extension of a given $\sigma$-finite measure. These sets are subsets of $H$-selectors for an appropriately chosen countable group $H$.

In the present paper we take a closer look at measurability properties of selectors. Putting a freeness assumption on the action of $G$ and assuming that $G$ is uncountable we prove that for a $\sigma$-finite measure one can always find a countable group $H$ such that no $H$-selector is measured by any invariant extension of the given measure. We show also that the situation for subgroups of full cardinality is just the opposite. Imposing a stronger freeness condition and

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assuming that the cardinality of $G$ has uncountable cofinality we prove that any
$\sigma$-finite ergodic measure admits an invariant extension which measures at least
one $H$-selector for any subgroup $H$ of full cardinality. This was conjectured for
Lebesgue measure and for Borel uncountable subgroups of the reals by Cichoń.

Now we set some notation. We write $H^Y = \{hx: x \in Y, h \in H\}$ for
$H \subset G$ and $Y \subset X$, and $h^Y = \{h\}^Y$ for $h \in G$, $Y \subset X$. $\mu_*$ and $\mu^*$
denote the inner and outer measure, respectively. We say that the action of $G$
is $\mu$-free if $\mu^*(\{x \in X: hx = x\}) = 0$ for any $h \in G \setminus \{e\}$ ($e$ is the identity of
$G$). Notice that the action of any subgroup of the group of all isometries of a
Euclidean space is $\mu$-free for any invariant extension of Lebesgue measure. The
action of $G$ is free if $\{x \in X: hx = x\} = \emptyset$ for any $h \in G \setminus \{e\}$. A measure
$\mu$ is called an invariant extension of $\mu$ if $\mu^*$ is an invariant measure, each $\mu$-
measurable set is $\mu^*$-measurable, and $\mu(Y) = \mu^*(Y)$ for any $\mu$-measurable set
$Y \subset X$. $\mu$ is called ergodic if for any two measurable sets $A, B \subset X$ with
$\mu(A) > 0$ and $\mu(B) > 0$ there is an $h \in G$ such that $\mu(A \cap hB) > 0$. Note
that Lebesgue measure, or more generally Haar measure on a locally compact
group, is ergodic. $|A|$ denotes the cardinality of $A$. $N$ stands for the set of
positive integers.

Our first theorem states that in the case of $\sigma$-finite measures one can always
find a countable subgroup $H$ such that $H$-selectors behave just like $Q$-selectors
where $Q$ is the group of the rationals on the real line.

**Theorem 1.** Let $G$ be uncountable, and let $\mu$ be $\sigma$-finite. Suppose $G$ acts $\mu$-
freely on $X$. Then there exists a countable subgroup $H$ of $G$ such that each
$H$-selector is nonmeasurable with respect to any invariant extension of $\mu$.

**Proof.** The terminology in this proof is from [6]. Since $\mu$ is $\sigma$-finite, one can
construct by transfinite induction using [6, Lemma 3.3] a countable family of $\mu$-
measurable sets $\{A_n: n \in N\}$ such that $\mu(X \setminus \bigcup_{n=1}^{\infty} A_n) = 0$ and
$A_n$ is infinitely covered by some countable $H_n$, $n \in N$. Let $H$ be the subgroup of $G$
generated by $\bigcup_{n=1}^{\infty} H_n$. Let $V$ be any $H$-selector, and let $\mu^*$ be an invariant extension of
$\mu$ for which $V$ is measurable. Since $HV = X$ and $H$ is countable, we have
$\mu^*(V) > 0$. But then $\mu^*(V \cap A_n) > 0$ for some $n \in N$, which contradicts [6,
Lemma 3.1]. □

Notice that the group $H$ in the above theorem may be very different from the
group of the rationals. For example, if $G$ is the group of all isometries of the
$n$-dimensional Euclidean space and $\mu$ is a $G$-invariant extension of Lebesgue
measure, then any countable infinite subgroup of $G$ consisting of orthogonal
linear transformations works. We can choose such a subgroup to be isomorphic
to a free group with countably many generators ($n \geq 3$) or to the infinite cyclic
group ($n \geq 2$).

Now we turn our attention to selectors of subgroups of higher cardinality.
We need some new notions. By an ideal on a set $Y$ we understand a family of
subsets of $Y$ not containing $Y$, closed under taking subsets and finite unions.
If $I$ is an ideal, then a family of subsets of $Y$ is called disjoint modulo $I$ if
the intersection of any two of its members is in $I$. Define $\text{sat}(I) = \min\{\kappa:\$
if $\mathcal{T}$ is a disjoint modulo $I$ family of subsets of $Y$ then $|\mathcal{T}| < \kappa\}$
and $\text{add}(I) = \min\{|\mathcal{T}|: \mathcal{T} \subset I \text{ and } \bigcup \mathcal{T} \notin I\}$. As usual $I$ is called a $\sigma$-ideal if
$\text{add}(I) > \omega$. Two ideals $I_1, I_2$ on $Y$ are called coherent if $A_1 \cup A_2 \neq Y$ for
any $A_1 \in I_1$ and $A_2 \in I_2$. If $I_1$ and $I_2$ are coherent, we denote by $[I_1, I_2]$ the ideal generated by $I_1$ and $I_2$. Clearly $\text{add}(I_1, I_2) \leq \min(\text{add}(I_1), \text{add}(I_2))$.

An ideal $I$ on $X$ is called invariant if for any $A \in I$ and $h \in G$ we have $hA \in I$. For any cardinal number $\lambda$ and any set $Y$ let $[Y]^{< \lambda}$ (resp. $[Y]^{\leq \lambda}$) denote $\{A \subseteq Y : |A| < \lambda\}$ (resp. $\{A \subseteq Y : |A| = \lambda\}$, $\{A \subseteq Y : |A| \leq \lambda\}$).

We identify ordinal numbers with the sets of their predecessors. For a cardinal number $\lambda$ let $\text{cf}(\lambda) = \min \{\kappa : \kappa$ is an ordinal and $\exists f : \kappa \rightarrow \lambda \lambda = \bigcup_{\alpha<\kappa} f(\alpha)\}$. A cardinal number $\kappa$ is called regular if $\text{cf}(\kappa) = \kappa$. For any cardinal $\lambda$, $\text{cf}(\lambda)$ is a regular cardinal.

**Lemma 1.** Let $I$ be an ideal on $Y$, and let $\kappa$ be a regular cardinal with $\kappa \leq \text{add}(I)$ and $\kappa < \text{sat}(I)$. Then there exists an ideal $J$ such that:

1. $J \supset I$;
2. $\text{add}(J) \geq \kappa$;
3. for each $A \notin I$ there is $B \in J \setminus I$ with $B \subseteq A$.

**Proof.** (The presented proof follows a suggestion of Blass which substantially simplifies the author's original argument.) Since $\kappa < \text{sat}(I)$, we can find a maximal disjoint modulo $I$ family of cardinality $\geq \kappa$. Denote this family $\mathcal{B}$.

Let $J = \{B \subseteq Y : \exists C \subseteq I \exists \mathcal{P} \subseteq [\mathcal{B}]^{< \kappa} B \subseteq C \cup \bigcup \mathcal{P}\}$.

Obviously $J$ fulfills (i). Since $\kappa$ is regular and $\kappa \leq \text{add}(I)$, $J$ fulfills (ii).

If $A \notin I$, then by maximality of $\mathcal{B}$ there exists $B \in \mathcal{B}$ with $A \cap B \notin I$.

Clearly $A \cap B \in J$, so (iii) is satisfied, too. □

Notice that by Ulam's theorem if $\text{add}(I)$ is a successor cardinal, then $\text{add}(I) < \text{sat}(I)$. In this case (ii) means simply $\text{add}(J) \geq \text{add}(I)$. Nevertheless in general the condition $\kappa < \text{sat}(I)$ cannot be dropped. For if $I$ and $J$ are as in the above lemma we have $\text{add}(J) < \text{sat}(I)$ because applying (iii) and (i) one can construct $\text{add}(J)$ pairwise disjoint sets outside of $I$.

In the sequel we will use only the following corollary of Lemma 1. This corollary can also be inferred from a much deeper result of Weglorz [9]. The author decided to present the direct proof here because of its simplicity.

**Corollary 1.** Let $\kappa$ be a cardinal, and let $I$ be an ideal on $Y$. Then there exists an ideal $J$ such that:

1. $J \supset I$;
2. $\text{add}(J) \geq \text{add}(I)$;
3. $\forall A \in [Y]^{< \kappa} \exists B \in J \cap [Y]^{< \kappa} B \subseteq A$.

**Proof.** If $[Y]^{< \kappa}$ is not contained in $I$, take $A \in [Y]^{< \kappa} \setminus I$ and define $J = \{B \subseteq Y : B \cap A \in I\}$. Then clearly (i), (ii), and (iii) are fulfilled. Assume that $[Y]^{< \kappa} \subseteq I$. If $\text{add}(I) > \kappa$ or $\text{sat}(I) \leq \kappa$, put $J = I$. Then (i) and (ii) are obviously satisfied. When $\text{add}(I) > \kappa$, we have $[Y]^{\kappa} \subseteq I$ as $[Y]^{< \kappa} \subseteq I$ and (iii) is fulfilled. When $\text{sat}(I) \leq \kappa$, (iii) is again true since each set from $[Y]^{\kappa}$ can be divided into $\kappa$ many pairwise disjoint sets from $[Y]^{\kappa}$. If $\text{sat}(I) > \kappa \geq \text{add}(I)$, notice that $\text{add}(I)$ is a regular cardinal and apply Lemma 1 (add(I) playing the role of the $\kappa$ in the lemma). As for (iii), by Lemma 1(iii) each set from $[Y]^{\kappa} \setminus I$ contains a set from $(I \setminus I) \cap [Y]^{< \kappa}$ and we have $J \cap [Y]^{< \kappa} \supset (I \setminus I) \cap [Y]^{< \kappa}$ since $[Y]^{< \kappa} \subseteq I$. □
Now we prove a lemma concerning extensions of invariant ideals. Our method of construction owes much to ideas of Kakutani and Oxtoby [5] and Hulanicki [4].

**Lemma 2.** Assume $G$ is uncountable and acts freely on $X$. Let $I$ be an invariant ideal on $X$. Then there exists an invariant ideal $J$ such that:

(i) $J \supset I$;

(ii) $\text{add}(J) \geq \min(\text{add}(I), \text{cf}(|G|))$;

(iii) $J$ contains an $H$-selector for each subgroup $H$ of $G$ with $|H| = |G|$.

**Proof.** Let $W$ be a $G$-selector. Put $\lambda = |G|$ and $\kappa = \text{cf}(\lambda)$. Let $\{G_\alpha : \alpha < \kappa\}$ be a family of subgroups of $G$ such that $G_\alpha \subset G_\beta$ for $\alpha < \beta < \kappa$, $|G_\alpha| < \lambda$, and $\bigcup_{\alpha < \kappa} G_\alpha = G$. For convenience we assume also that $G_\alpha \setminus \bigcup_{\xi < \alpha} G_\xi \neq \emptyset$.

Let $X_\alpha = (G_\alpha \setminus \bigcup_{\xi < \alpha} G_\xi)W$. We define an ideal on $\kappa$ as

$$I' = \left\{ D \subset \kappa : \bigcup_{\alpha \in D} X_\alpha \in I \right\}.$$ 

First we show that $I'$ is coherent with $[\kappa]^{<\kappa}$. Take $D \in [\kappa]^{<\kappa}$. Since $\kappa$ is regular, we can find $\beta < \kappa$, which is greater than all elements of $D$. Take $h \in G_\beta \setminus \bigcup_{\xi < \beta} G_\xi$. Since $G$ acts freely and $W$ is a $G$-selector, $h(\bigcup_{\alpha \in D} X_\alpha) \cap \bigcup_{\alpha \in D} X_\alpha = \emptyset$, i.e., $h(\bigcup_{\alpha \in \kappa \setminus D} X_\alpha) \cup \bigcup_{\alpha \in \kappa \setminus D} X_\alpha = X$. Thus $\kappa \setminus D \not\in I'$ as $I$ is invariant.

Put $\overline{I} = \{I', [\kappa]^{<\kappa}\}$. Then $\text{add}(\overline{I}) \geq \min(\text{add}(I), \kappa)$. Let $\overline{J}$ be an ideal on $\kappa$ extending $\overline{I}$ whose existence is guaranteed by Corollary 1. Let

$$J' = \left\{ A \subset X : \exists D \in \overline{J} A \subset \bigcup_{\alpha \in D} X_\alpha \right\}.$$ 

$J'$ is invariant. Let $h \in G$. Then $h \in G_\beta$ for some $\beta < \kappa$. It is enough to check that $hA \in J'$ for $A$ of the form $\bigcup_{\alpha \in D} X_\alpha$ for some $D \in \overline{J}$. But then $hA \setminus A \subset \bigcup_{\alpha < \beta} X_\alpha \in J'$ since $\{\alpha : \alpha < \beta\} \in \overline{J}$. Notice that $J'$ and $I$ are coherent. Otherwise there are $A_1 \in I$, $A_2 \in J'$ such that $A_1 \cup A_2 = X$. Then there is $D \in \overline{J}$ such that $A_2 \subset \bigcup_{\alpha \in D} X_\alpha$. Thus $\bigcup_{\alpha \in \kappa \setminus D} X_\alpha \subset A_1$ whence $\kappa \setminus D \in I$. But $\overline{I} \subset \overline{J}$ and thus $\kappa \setminus D \in \overline{J}$, a contradiction.

Let $J = [J', I]$. Clearly $J$ is invariant and $J \supset I$. Since $\text{add}(J') \geq \text{add}(\overline{I}) \geq \text{add}(I') \geq \min(\text{add}(I), \kappa)$, we have $\text{add}(J) \geq \min(\text{add}(I), \kappa)$. Thus (i) and (ii) are fulfilled. Now we show (iii). Let $H$ be a subgroup of $G$ with $|H| = \lambda$. Put $D = \{\alpha < \kappa : H \cap G_\alpha \setminus \bigcup_{\xi < \alpha} G_\xi \neq \emptyset\}$. Then $D \in [\kappa]^{<\kappa}$, so there is $D' \subset \bigcup_{\alpha \in D} X_\alpha$. There exist $y \in W$ and $h \in G$ with $x = hy$. We can also find $\beta \in D'$ such that $h \in G_\alpha$ for some $\alpha < \beta$. Then $\text{add}(J') \geq \text{add}(\overline{I}) \geq \text{add}(I') \geq \min(\text{add}(I), \kappa)$, we have $\text{add}(J) \geq \min(\text{add}(I), \kappa)$. Thus (i) and (ii) are fulfilled. Now we show (iii). Let $H$ be a subgroup of $G$ with $|H| = \lambda$. Put $D = \{\alpha < \kappa : H \cap G_\alpha \setminus \bigcup_{\xi < \alpha} G_\xi \neq \emptyset\}$. Then $D \in [\kappa]^{<\kappa}$, so there is $D' \subset \bigcup_{\alpha \in D} X_\alpha$. There exist $y \in W$ and $h \in G$ with $x = hy$. We can also find $\beta \in D'$ such that $h \in G_\alpha$ for some $\alpha < \beta$. Then $\text{add}(J') \geq \text{add}(\overline{I}) \geq \text{add}(I') \geq \min(\text{add}(I), \kappa)$, we have $\text{add}(J) \geq \min(\text{add}(I), \kappa)$.

The following lemma is essentially due to Szpilrajn [7, §2].

**Lemma 3 (Szpilrajn).** Let $\mu$ be an invariant measure on $X$, and let $J$ be an invariant $\sigma$-ideal on $X$ such that $\mu_*(A) = 0$ for $A \in J$. Then there exists an
invariant extension of $\mu$ defined on the $\sigma$-algebra generated by the $\sigma$-algebra of $\mu$-measurable sets and $J$.

The next theorem shows that under certain assumptions a subgroup of full cardinality with properties like those in Theorem 1 cannot be constructed.

**Theorem 2.** Assume $\text{cf}(|G|) > \omega$. Suppose also that $G$ acts freely on $X$. Let $\mu$ be $\sigma$-finite and ergodic. Then there exists an invariant extension $\mu$ of $\mu$ such that for each subgroup $H$ of $G$ with $|H| = |G|$ there is a $\mu$-measurable $H$-selector.

**Proof.** Consider $I_\mu$ the invariant $\sigma$-ideal of $\mu$-measure 0 sets. Let $J$ be an ideal extending $I_\mu$ produced in Lemma 2. As $\text{add}(J) \geq \min(\text{add}(I_\mu), \text{cf}(|G|)) > \omega$, $J$ is a $\sigma$-ideal. Now we show that the assumption of Lemma 3 is fulfilled. Suppose $\mu_*(A) > 0$ for some $A \in J$. As $J$ is closed under taking subsets, we can assume that $A$ is $\mu$-measurable and $\mu(A) > 0$. Using the $\sigma$-finiteness and the ergodicity of $\mu$ we find a countable set $K \subseteq G$ with $\mu(X \setminus KA) = 0$, i.e., $X \setminus KA \in I_\mu \subseteq J$. As $KA \in J$ we get a contradiction. Thus $A \notin J$. Now Lemma 3 yields an invariant extension $\mu$ of $\mu$ for which all sets from $J$ are measurable. In particular, for each subgroup $H$ of cardinality $|G|$ there is a $\mu$-measurable $H$-selector. □

Since $\text{cf}(2^\omega) > \omega$, Theorem 2 gives the following corollary. (The “if” direction of the second part of the corollary can be shown by the same argument as in the standard proof that any $Q$-selector, where $Q$ denotes the rationals, is not Lebesgue measurable.)

**Corollary 2.** There exists a translation invariant extension of Lebesgue measure on the real line which measures at least one $H$-selector for each group of translations $H$ with $|H| = 2^\omega$. In particular, assuming the Continuum Hypothesis, each $H$-selector of a subgroup $H$ of the reals is nonmeasurable with respect to any invariant extension of Lebesgue measure if and only if $H$ is countable and dense.

In the context of Theorems 1 and 2 the following question seems to be interesting. Let $G$ act freely on $X$, and let $\mu$ be invariant, $\sigma$-finite, and ergodic. Does there exist an invariant extension of $\mu$ which measures at least one $H$-selector for each uncountable subgroup $H$ of $G$? The author does not know the answer even for Lebesgue measure on the real line (without assuming the Continuum Hypothesis of course).

**References**


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