INvolutions Fixing Products Of Circles

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In memory of E. E. Floyd

Abstract. This paper determines the possible equivariant bordism classes of involutions having fixed set a union of products of circles.

1. Introduction

Consider the involution on the projective plane $RP^2$ defined by

$$T([x_0, x_1, x_2]) = [-x_0, x_1, x_2].$$

The fixed point set of this involution consists of a point, $[1, 0, 0]$, with trivial normal bundle and a circle, $S^1$ or $RP^1$, given by the points with $x_0 = 0$, with normal bundle the nontrivial line bundle $\xi$ over $RP^1$.

Forming the product of $m$-copies of this example, one obtains an involution on $(RP^2)^m = RP^2 \times \cdots \times RP^2$ given by $T \times \cdots \times T$, for which the fixed point set is the union of $\binom{m}{k}$ copies of $(RP^1)^k$ with normal bundle

$$\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_k \oplus (2m - 2k) \rightarrow RP^1 \times \cdots \times RP^1$$

for $0 \leq k \leq m$. Here $\binom{m}{k}$ is the binomial coefficient, $\xi_i$ is the line bundle over the $i$th factor, and $(RP^1)^0$ is interpreted as being a point.

In their book [2] Conner and Floyd proved that, up to bordism, $(RP^2, T)$ is the only involution with fixed set the union of a point and a circle. (See [2, (27.6)].) The purpose of this note is to establish the generalization:

Theorem. If $(M^n, T)$ is an involution having fixed point set a union of copies of $(RP^1)^k$, with $0 \leq k \leq n$, then either $(M^n, T)$ bounds or $n = 2m$ and $(M^n, T)$ is equivariantly cobordant to the involution $((RP^2)^m, T \times \cdots \times T)$.

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2. The Proof

From [2, (28.1)], one has an exact sequence

$$0 \rightarrow \mathcal{N}_n^{Z_2} F \bigoplus_{j=0}^{n} \mathcal{N}_{n-1}(BO_j) \rightarrow \mathcal{N}_{n-1}(RP^\infty) \rightarrow 0,$$
which shows that the cobordism class of an involution is determined by the bordism class of its fixed set and normal bundle. To study involutions fixing unions of products of $RP^1$'s one needs

**Lemma.** If $\eta^j \to (RP^1)^k$ is a $j$-plane bundle over the $k$-fold product which is nonbounding (in $\mathscr{N}_k(BO_j)$), then $j \geq k$ and $((RP^1)^k, \eta^j)$ is bordant to the bundle

$$\xi_1 \oplus \cdots \oplus \xi_k \oplus (j-k) \to (RP^1)^k.$$  

**Proof.** The bordism class of a bundle is determined by its Stiefel-Whitney numbers, and since $(RP^1)^k$ is parallelizable, the only Stiefel-Whitney numbers that can possibly be nonzero are those of the form

$$w_1 w_2 \cdots w_i [(RP^1)^k],$$

where $w_i = w_i(\eta)$ and $i_1 \leq i_2 \leq \cdots \leq i_r$, $i_1 + \cdots + i_r = k$. Further, if $x \in H^*((RP^1)^k; \mathbb{Z}_2)$, then $x^2 = 0$, and one may suppose $i_1 < i_2 < \cdots < i_r$. From Wu’s theorem [3]

$$s^i w_i = \sum_{u=0}^s \binom{t-s-1+u}{u} w_{s-u} w_{s+u}$$

and triviality of the action of the Steenrod algebra in $H^*((RP^1)^k; \mathbb{Z}_2)$, one has

$$0 = S^q w_{2^{e+1}b} = w_{2^{e+1}b} w_{2^a} + w_{2^{e+1}b + 2^a}.$$ 

Thus, the only Stiefel-Whitney numbers $w_i \cdots w_r [(RP^1)^k]$ with $i_1 + \cdots + i_r = k$ which can be nonzero are those in which the $i$'s have no common powers of 2 in their dyadic expansion, and these are nonzero if and only if $w_k [(RP^1)^k]$ is nonzero. If $((RP^1)^k, \eta^j)$ is nonbounding, $w_k (\eta^j) \neq 0$, so $j \geq k$ and $\eta$ is bordant to $\xi_1 \oplus \cdots \oplus \xi_k \oplus (j-k)$, which also has $w_k$ nonzero.  □

The proof of the theorem is now a very easy inductive argument. One considers a class

$$\alpha = ((RP^1)^{k_1}, \eta^{n-k_1}) \cup \cdots \cup ((RP^1)^{k_r}, \eta^{n-k_r})$$

in $\sum_{k=0}^n \mathcal{N}_k(BO_{n-k})$, where $k_1 < k_2 < \cdots < k_r$ and each bundle $((RP^1)^{k}, \eta^{n-k})$ is nonbounding. One can suppose $\eta^{n-k} = \xi_1 \oplus \cdots \oplus \xi_k \oplus (n-2k)$ with no loss, and hence $n \geq 2k_r$.

The hypothesis for the induction on $k_r$ is that $\alpha$ is the fixed data of an involution $(M^n, T)$ if and only if $n = 2k_r$ and the $k_i$ occurring are precisely those for which $(k_i)$ is nonzero mod 2.

The case $k_r = 0$ is trivial. One is asking that a point with trivial $n$-plane bundle be the fixed set of an involution $(M^n, T)$. This can only happen for $n = 0$ with the trivial involution on a point, i.e., $((RP^2)^m, T \times \cdots \times T)$ with $m = 0$. (See [2, remark following (25.1)].) Now consider the class $\alpha$ and suppose $\alpha$ is the fixed set of an involution $(M^n, T)$. From [2, (26.4)], one has a commutative diagram

$$\bigoplus_{k=0}^n \mathcal{N}_k(BO_{n-k}) \xrightarrow{\partial} \mathcal{N}_{n-1}(RP^\infty)$$

and

$$\bigoplus_{k=0}^{n-1} \mathcal{N}_k(BO_{n-1-k}) \xrightarrow{\partial} \mathcal{N}_{n-2}(RP^\infty)$$

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where \( \bigoplus 1 \) adds a trivial line bundle and \( \Delta \) is the Smith homomorphism.

Clearly, if \( n > 2k_r \), then one has \( \alpha = (\bigoplus 1)^{n-2k_r} \alpha' \), where

\[
\alpha' = ((RP^1)^{k_1}, \eta^{2k_r-k_1}) \cup \cdots \cup ((RP^1)^{k_r}, \eta^{k_r})
\]

has the same sequence \( k_1 < \cdots < k_r \) as does \( \alpha \). Then

\[
\partial \alpha' = \Delta^{n-2k_r} \partial (\bigoplus 1)^{n-2k_r} \alpha' = \Delta^{n-2k_r} \partial \alpha = \Delta^{n-2k_r} 0 = 0,
\]

so \( \alpha' \) is the fixed set of an involution on a manifold of dimension \( 2k_r \). If \( n = 2k_r \), \( \alpha' = \alpha \) and nothing has been done so far. Say \( \alpha' = F(N^{2k_r}, S) \).

Now consider \( \beta = F((N^{2k_r}, S) \cup ((RP^2)^{k_r}, T \times \cdots \times T)) \). This is the fixed data of an involution with

\[
\beta = ((RP^1)^{j_1}, \eta^{2k_r-j_1}) \cup \cdots \cup ((RP^1)^{j_r}, \eta^{2k_r-j_r})
\]

having \( j_s < k_r \) since the dimension \( k_r \) components of the fixed sets in \((RP^2)^{k_r}\) and \( N^{2k_r} \) cancelled out. By the inductive hypothesis, this can only happen if \( \beta = 0 \), for the dimension of the involution exceeds \( 2j_s \).

Thus \( \alpha' = F((RP^2)^{k_r}, T \times \cdots \times T) \) is the fixed set of the standard involution, or \( k_i \) occurs precisely when \( \binom{k_r}{k_i} \) is odd.

Now assume \( n > 2k_r \). Then

\[
\partial (\bigoplus 1) \alpha' = \Delta^{2k_r-1} \partial \alpha = 0.
\]

Thus \( (\bigoplus 1) \alpha' \) is the fixed set of an involution. However, [2, (24.2)] observes that the real projective space bundle of \((\bigoplus 1) \alpha' \) is cobordant to \((RP^2)^{k_r}\), which is not a boundary, while the projective space bundle of the fixed set of an involution bounds [2, (24.1)]. This is a contradiction.

This completes the induction and the proof of the theorem.

3. Bundles over \((RP^1)^k\)

The most direct approach to proving the theorem would start by finding the possible Stiefel-Whitney classes for all vector bundles over \((RP^1)^k\). Unfortunately, the classes turn out to be surprisingly complicated, and the argument was chosen to bypass this point. It seems desirable to describe the classes.

**Proposition.** Let \( H^*((RP^1)^k; Z_2) = Z_2[\alpha_1, \ldots, \alpha_k]/(\alpha_i^2 = 0) \), where \( \alpha_i \) is the 1-dimensional class given by projection on the \( i \)th factor. There are vector bundles over \((RP^1)^k\) having Stiefel-Whitney classes

\[
\begin{align*}
(1) & \quad 1 + \alpha_i; \\
(2) & \quad 1 + \alpha_i \alpha_j \alpha_k, \ i_1 < i_2; \\
(3) & \quad 1 + \alpha_i \alpha_j \alpha_k \alpha_\ell, \ i_1 < i_2 < i_3 < i_4; \ \text{and} \\
(4) & \quad 1 + \alpha_i \alpha_\ell \alpha_\ell \cdots \alpha_\ell, \ i_1 < i_2 < \cdots < i_\ell.
\end{align*}
\]

and every bundle over \((RP^1)^k\) has Stiefel-Whitney class a product \( \prod (1 + x_j) \) for some subset of this set of classes.

**Proof.** To construct the given classes, let \( r = 1, 2, 4, \) or 8 and consider the projection \( \pi: (RP^1)^k \to (RP^1)^r \) corresponding to \( \alpha_i, \ldots, \alpha_i, \). Compose this with a degree one map to the sphere \( S^r \) and pull back the \( r \)-plane bundle over the sphere having \( w = 1 + \sigma_r \).
Being given any vector bundle \( \eta \) over \((\mathbb{RP}^1)^k\), one can choose a unique sum \( \sum \rho \) of these bundles for which \( w_i(\sum \rho) = w_i(\eta) \) for \( i \leq 8 \). Then, for \( \theta = \eta - \sum \rho \),

\[
w(\theta) = 1 + w_{2s} + \text{higher terms}
\]

with \( s > 3 \).

If one considers the Thom space of \( \theta \) with Thom class \( U \), one has

\[
Sq^{2s}U = w_{2s}U, \quad Sq^iU = w_iU = 0 \quad \text{for } 1 \leq i < 2s.
\]

From [1], there are secondary cohomology operations \( p^{2s-j}, 0 < j < 2s \), with

\[
w_{2s}U = Sq^{2s}U = \sum Sq^j p^{2s-j}U = \sum Sq^j (y_{2s-j}U)
\]

\[
= \sum y_{2s-j} Sq^j U = \left( \sum y_{2s-j} w_j \right) U = 0
\]

since Steenrod operations are trivial in \((\mathbb{RP}^1)^k\). Thus, \( w_{2s}(\theta) = 0 \) and so \( w(\theta) = 1 \). \( \square \)

The most obvious vector bundles over \((\mathbb{RP}^1)^k\) are the line bundles, giving Stiefel-Whitney classes \( 1 + x \) for every 1-dimensional class \( x \). It is immediate that

\[
1 + \alpha \beta = (1 + \alpha + \beta)(1 + \alpha)(1 + \beta),
\]

so the classes of 2-plane bundles described above can be given by sums of line bundles.

Surprisingly, the classes of the 4-plane and 8-plane bundles cannot be expressed as sums of line bundles. One has

\[
(1 + \alpha_{i_1} + \cdots + \alpha_{i_s}) = (1 + \alpha_{i_1}) \cdots (1 + \alpha_{i_s}) \prod_{u < t} (1 + \alpha_u \alpha_{i_t}),
\]

as can readily be seen by induction on \( s \), and so the classes of line bundles are all obtained using only the first two types. Thus, sums of line bundles do not give all Steifel-Whitney classes.

REFERENCES


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