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DIRECTIONS ON A CURVE 

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Abstract. We obtain a sharp $L^2$ estimate for the maximal operator associated 
with uniformly distributed directions on a curve of finite type in $\mathbb{R}^n$.

INTRODUCTION

Let $\gamma : [0, 1] \to S^{n-1}$ be a smooth curve crossing each hyperplane of $\mathbb{R}^n$ a 
finite number of times. If $N$ denotes the family of all cylinders in $\mathbb{R}^n$ having 
eccentricity $N$ and direction in $\gamma$, it is proved in [C] that the maximal operator 

$$M_N f(x) = \sup_{x \in \mathcal{B} \in N} \frac{1}{|R|} \int_R |f(y)| \, dy$$

satisfies the estimate 

$$(1) \quad \|M_N f\|_{L^2} \leq C_\gamma (\log N)^2 \|f\|_{L^2}$$

where $C_\gamma$ is independent of $N$.

The purpose of this note is to show that by imposing an additional condition on $\gamma$ one can prove a stronger result.

Let $\gamma$ be a smooth curve satisfying 

$$(*) \quad \text{For all } t \in [0, 1], \text{ the set } \{\gamma^{(j)}(t)\}_{0 \leq j < \infty} \text{ spans } \mathbb{R}^n.$$

For a positive integer $m$ let $\mathcal{B}_m$ denote the family of all cylinders in $\mathbb{R}^n$ 
pointing in the direction of $\gamma(j/2^m)$ for some $0 \leq j \leq 2^m$. Let $M_m f(x) = \sup_{x \in \mathcal{B} \in \mathcal{B}_m} (1/|R|) \int_R |f(y)| \, dy$. Then we will prove the following:

Theorem. If $\gamma$ satisfies $(*)$ then 

$$(2) \quad \|M_m f\|_{L^2} \leq C_m \|f\|_{L^2}$$

where $C_m$ is independent of $m$.

If $n = 2$ or if $\gamma$ is contained in a 2-dimensional subspace, (2) is known to be 
true (see [S] or [B]). Also, since $M_m$ dominates $M_{2^m}$, (2) implies an improved 
version of (1).

In what follows all the constants are independent of $m$.

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AUXILIARY LEMMAS

We will now prove some consequences of (*) that will be used to prove the theorem.

A simple compactness argument shows that if $\gamma$ satisfies (*), then there exist an integer $L$ and $c > 0$ such that for all $\xi \in S^{n-1}$ and $t \in [0, 1]$

$$\sum_{i=0}^{L} |\xi \cdot \gamma(i)(t)| \geq c.$$

For $j = 0, 1, 2$ let $\mathcal{U}_j = \{ (\xi, t) \in S^{n-1} \times [0, 1] : |\xi \cdot \gamma(i)(t)| \leq c2^{-(l+1)} \text{ for } l \leq j \}$. Then we have

**Lemma 1.** There exist $\delta_j > 0$ and $c_j > 0$ such that for all $(\xi, t) \in \mathcal{U}_j$

$$|s - t| < \delta_j \Rightarrow |\xi \cdot (\gamma(j)(s) - \gamma(j)(t))| \geq c_j |s - t|^{L-j}.$$

**Proof.** If the lemma is false, we can find sequences $e_k \to 0$, $\delta_k \to 0$, $(\xi_k, t_k) \in S^{n-1} \times [0, 1]$, and $s_k$ such that $|s_k - t_k| < \delta_k$ and

$$|\xi_k \cdot (\gamma(j)(s_k) - \gamma(j)(t_k))| < e_k |s_k - t_k|^{L-j}.$$

Since $\mathcal{U}_j$ is compact, by passing to a subsequence, we can assume that $(\xi_k, t_k)$ converges to $(\xi, t) \in \mathcal{U}_j$. By Taylor's theorem (5) implies that $\xi \cdot \gamma(i)(t) = 0$ for $l = j + 1, \ldots, L$. This contradicts (3).

Lemma 1 implies that there exist integers $N_j (\sim \delta_j^{-1})$ such that for all $\xi$ in $S^{n-1}$ the function $\xi \cdot \gamma(j)(t)$ has at most $N_j$ zeros on $\{ t \in [0, 1] : |\xi \cdot \gamma(i)(t)| < c2^{-(l+1)} \text{ for } 0 \leq l \leq j \}$. For $\xi \in S^{n-1}$ let $\nu_\xi(t) = \xi \cdot \gamma(t)$, $\mathcal{V}_\xi^1 = \{ t \in [0, 1] : |\nu_\xi(t)| > c/2 \}$, and $\mathcal{V}_\xi^2 = \{ t \in [0, 1] : |\nu_\xi(t)| < c/2 \text{ and } |\nu_\xi''(t)| > c/4 \}$. Since $\mathcal{V}_\xi^1$ and $\mathcal{V}_\xi^2$ are open (in $[0, 1]$) and disjoint, we can write each $\mathcal{V}_\xi^j$ as a countable union of disjoint intervals. Since between each two intervals of $\mathcal{V}_\xi^1$ there exists a $t$ for which either $\nu_\xi(t) = 0$ or $\nu_\xi''(t) = 0$, Lemma 1 implies that $\mathcal{V}_\xi^1$ is the union of at most $N_0 + N_1$ (independent of $\xi$) intervals. A similar argument applied to $\mathcal{V}_\xi^2$ in the complement of $\mathcal{V}_\xi^1$ together with the fact that, on the complement of $\mathcal{V}_\xi^1 \cup \mathcal{V}_\xi^2$, $\nu_\xi''(t)$ has at most $N_2$ zeros shows that the complement of $\mathcal{V}_\xi^1 \cup \mathcal{V}_\xi^2$ can be written as a union of no more than $2(N_0 + N_1 + N_2)$ closed intervals where, on each of these, $\nu_\xi(t)$ is monotonic. Let $I = [a, b]$ be one such interval, and let $t_0 \in [a, b]$ be such that $|\nu_\xi''(t)| = \min_I |\nu_\xi''(t)|$. Then we have

$$\nu_\xi(t) = \sum_{j=0}^{L-1} \frac{\nu_\xi^{(j+1)}(t_0)}{j!} (t - t_0)^j + R_{t_0}(t)$$

$$= p_\xi(t) + R_{t_0}(t) \quad \text{where } |R_{t_0}(t)| \leq C|t - t_0|^L.$$

Thus if $\delta_y = 1/2 \min \{ \min_j \delta_j, c_1 C^{-1} \}$ we have for $|t - t_0| < \delta_y$ and $t \neq t_0$

$$\left| \frac{p_\xi(t)}{\nu_\xi(t)} - 1 \right| \leq \frac{C|t - t_0|^L}{|\nu_\xi(t)|} \leq \frac{1}{2}.$$
which implies

\[ |p'\xi(t)| \leq |v'\xi(t)| \leq 2|p'\xi(t)|. \]

If we let \( p'\xi(t) - v'\xi(t_0) = q_\xi(t) \), we have by Lemma 1 that there exist \( c_\xi > 0 \) such that \( |q_\xi(t)| \approx c_\xi|t - t_0|^k \) for \( |t - t_0| < \delta_\gamma \) and for some \( k \) with \( 1 \leq k \leq L - 1 \). If \( |t - t_0| > \delta_\gamma \) and \( t \in I \), Lemma 1 implies that \( |v'\xi(t)| \geq c_1\delta_\gamma L^k - 1 \). Since \( v_\xi(t) \) has at most two zeros on \( I \), we can divide \( \{ t \in I : |t - t_0| < \delta_\gamma \} \) in no more than four intervals where \( v_\xi(t) \) is monotonic and of constant sign satisfying estimates like the above. Thus, if we let \( N_\gamma = 10(N_0 + N_1 + N_2) \) and \( c_\gamma = \min\{c/4, c_1\delta_\gamma L^k - 1\} \), we obtain

**Lemma 2.** There exist an integer \( N_\gamma \) and \( c_\gamma > 0 \) such that for all \( \xi \) in \( S^{n-1} \) we have

\[ [0, 1] = U_1^{c_\xi} \cup \cdots \cup U_N^{c_\xi} \cup V_1^{c_\gamma} \cup \cdots \cup V_{M_\xi}^{c_\gamma} \cup W_1^{c_\gamma} \cup \cdots \cup W_K^{c_\gamma}. \]

where \( N_\xi + M_\xi + K_\xi \leq N_\gamma \) and where the \( U_i^{c_\xi} \)'s, \( V_i^{c_\gamma} \)'s, and \( W_i^{c_\gamma} \)'s are closed intervals with disjoint interiors for which

(i) \( |v_\xi(t)| \geq c_\gamma \) on \( \cup_i U_i^{c_\xi} \),

(ii) \( |v'\xi(t)| \geq c_\gamma \) on \( \cup_i V_i^{c_\gamma} \), and

(iii) for each \( i \leq K_\xi \) there exist \( c_i^{c_\gamma} > 0 \), \( t_0 \in W_i^{c_\gamma} \), and \( k = k_\xi, i, t_0 \) with \( 1 \leq k \leq L \) such that

\[ |v_\xi(t)| \approx |v_\xi(t_0)| + c_i^{c_\gamma}|t - t_0|^k \quad \text{and} \quad |v_\xi(t)| \approx c_i^{c_\gamma}|t - t_0|^k. \]

**Proof of Theorem.** The proof is based in a square function argument following the ideas in [W, NSW].

Let \( \varphi \in C^\infty_c(\mathbb{R}) \) be nonnegative, with \( \varphi \equiv 1 \) on \([-\frac{1}{4}, \frac{1}{4}] \) and such that \( \int_{-\infty}^{\infty} \varphi(t)\, dt = 1 \). For \( h > 0 \) let \( \varphi_h(t) = h^{-1}\varphi(h^{-1}t) \) and let \( \omega_j = \gamma(j2^{-m}) \).

For \( 0 \leq j \leq 2^m \) let

\[ T_{h,j}^m f(x) = \int_{-\infty}^{\infty} f(x - t\omega_j)\varphi_h(t)\, dt, \quad T^m f(x) = \sup_{h,j} |T_{h,j}^m f(x)|. \]

Then a simple geometric argument shows that it suffices to prove that

\[ \|T^m f\|_{L^2} \leq C_\gamma m\|f\|_{L^2} \quad \text{for} \quad f \geq 0. \]  

For \( m = 1 \), (6) follows from the boundedness of the one-dimensional Hardy-Littlewood maximal operator. Suppose (6) is true for \( m = 1 \). Then for \( f \geq 0 \)

\[ T^m f(x) \leq T^{m-1} f(x) + \sup_{h,j} |T_{h,2j}^m f(x) - T_{h,2j-1}^m f(x)| \]

\[ = T^m f(x) + \sup_{h,j} H_{h,j}^m f(x) \]

and (6) will follow if we can show that

\[ \left\| \sup_{h,j} H_{h,j}^m f \right\|_{L^2} \leq C_\gamma \|f\|_{L^2}. \]

For \( j = 1, \ldots, 2^m \), let \( \Gamma_j \) be the cone \( \{ \xi \in \mathbb{R}^n : |\xi \cdot \omega_j| > c_1 2^{-mL}|\xi| \} \) and let \( K_j \) be the complement of \( \Gamma_j \), and for \( j = 1, \ldots, 2^m-1 \), let \( f(x) = f_j(x) + r_j(x) \)

where \( f_j(x) = \chi_{K_j} \hat{f}(\xi) \). The Fourier transform. [License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use]
An argument similar to the one in [W, p. 88] shows that

$$\sup_{h, j} H_{h, j}^m f(x) \leq C(g_1(f)(x) + g_2(f)(x))$$

where

$$g_1(f)(x) = \left( \sum_{j=1}^{2^{m-1}} \left( \sup_h T_{h, 2j}^m |r_j(x)| + \sup_h T_{h, 2j-1}^m |r_j(x)| \right) \right)^{1/2},$$

$$g_2(f)(x) = \left( \int_0^\infty \sum_{j=1}^{2^{m-1}} \{T_{h, 2j}^m f_j(x) - T_{h, 2j-1}^m f_j(x)\}^2 \frac{dh}{h} \right)^{1/2}.$$

Thus (8) and hence the theorem will be a consequence of the following two estimates:

(9) \[ \|g_1(f)\|_{L^2} \leq C \gamma \|f\|_{L^2}, \]

(10) \[ \|g_2(f)\|_{L^2} \leq C \gamma \|f\|_{L^2}. \]

Proof of (9). By the boundedness of the one-dimensional Hardy-Littlewood maximal operator and Plancherel's theorem, one has

(11) \[ \|g_1(f)\|_{L^2}^2 \leq C \int_{\mathbb{R}^n} \sum_{j=1}^{2^{m-1}} |r_j(x)|^2 \, dx = C \int_{\mathbb{R}^n} \sum_{j=1}^{2^{m-1}} \chi_{K_j \cup K_{j-1}}(\xi) \hat{f}(\xi)|^2 \, d\xi. \]

Since the $K_j$'s are conic, it is enough to prove that no $\xi \in S^{n-1}$ belongs to more than $C_\gamma$ of the $K_j$'s. Given $\xi$ in $K_{j_0}$ we have $|v_\xi(t)| \leq c_1 2^{-mL}$. By (4), if $k > c_1^{1/L}$, then $\xi$ does not belong to $K_{j_0 \pm k}$. Thus $\xi$ does not belong to more than $N_\gamma c_1^{1/L}$ of the $K_j$'s.

Proof of (10). Plancherel's theorem implies that

(12) \[ \|g_2(f)\|_{L^2}^2 = \int_{\mathbb{R}^n} \sum_{j=1}^{2^{m-1}} \int_0^\infty |\phi(h \xi \cdot \omega_{2j}) - \phi(h \xi \cdot \omega_{2j-1})|^2 |\hat{f}_j(\xi)|^2 \frac{dh}{h} \, d\xi, \]

where

(13) \[ m(\xi) = \sum_{j=1}^{2^{m-1}} \int_0^\infty |\phi(h \xi \cdot \omega_{2j}) - \phi(h \xi \cdot \omega_{2j-1})|^2 \chi_{\Gamma_{2j} \cap \Gamma_{2j-1}}(\xi) \frac{dh}{h}, \]

and we are left to prove that $m(\xi) \leq C_\gamma$. This is accomplished by dividing the curve $\gamma$ in pieces where one has control over the decay of $\phi(h \xi \cdot \omega_j)$ in $\xi$ and $j$ in estimating (13). The details are below.

By Lemma 2 we can, for each $\xi$, split the sum in (13) in no more than $N_\gamma$ sums of the form $\sum_{j_2 - m \in U_\xi}$, $\sum_{j_2 - m \in V_\xi}$, and $\sum_{j_2 - m \in W_\xi}$. Thus the theorem will follow if we can show that each of these sums is bounded with bound independent of $m$ and $\xi$. By homogeneity we only need to consider $\xi \in S^{n-1}$.
Since $|\xi \cdot \omega_j| \geq c_7$ for $j 2^{-m} \in U_7$, and since $\phi$ is a Schwartz function, we have that $|\hat{\phi}(h \xi \cdot \omega_{j+1}) - \hat{\phi}(h \xi \cdot \omega_j)|^2 \leq Ch^2|\omega_{j+1} - \omega_j|^2|\phi'(h \xi \cdot u_j)||$ with $|\xi \cdot u_j| \geq c_7$. This implies

$$\sum_{j 2^{-m} \in U_7} \int_0^\infty |\hat{\phi}(h \xi \cdot \omega_{j+1}) - \hat{\phi}(h \xi \cdot \omega_j)|^2 \frac{dh}{h} \leq C_7.$$

We now prove a similar estimate for $W_7^i$. There is no lack of generality in assuming that $W_7^i = [0, \varepsilon]$ and that $\psi(0) = 0$. Lemma 2 implies that for $j 2^{-m} \in W_7^i$

$$|\xi \cdot (\omega_{2j} - \omega_{2j-1})| \leq C_7 \xi \cdot \omega_j \leq C_7 \xi \cdot \omega_j.$$

$$|\xi \cdot (\omega_{2j} - \omega_{2j-1})| \leq C_7 \xi \cdot \omega_j \leq C_7 \xi \cdot \omega_j.$$

Since $\phi$ is smooth and rapidly decreasing, by (15) and (16) we obtain

$$\phi(h \xi \cdot \omega_{2j}) - \phi(h \xi \cdot \omega_{2j-1}) \leq C_7 \xi \cdot \omega_{2j} \leq C_7 \xi \cdot \omega_{2j}.$$

$$|\xi \cdot \omega_j| \geq C_7 \xi \cdot \omega_j \leq C_7 \xi \cdot \omega_j.$$

Splitting each integral in $\int_0^{a_j} + \int_a^\infty$ where the $a_j$'s are to be determined later and using (17) and (18) on each integral respectively we obtain that $\sum_{j 2^{-m} \in W_7^i}$ is dominated by

$$\sum_{j 2^{-m} \in W_7^i} C_7 \xi \cdot (\omega_{2j} - \omega_{2j-1}) \leq C_7 \xi \cdot \omega_{2j} \leq C_7 \xi \cdot \omega_{2j}.$$

To finish, put $\beta = k - \frac{1}{4}$, $\alpha = 3$, and let $\alpha_j = c_7^{-1} j^{-\beta} 2^{mk}$ in (19) obtaining

$$\sum_{j 2^{-m} \in W_7^i} \frac{2^{m+1}}{1} \leq C \sum_{1}^{2^{(k-\beta-1)}} j^{2\alpha(\beta-k)} \leq C \sum_{1}^{\infty} j^{-3/2}.$$

The terms $\sum_{j 2^{-m} \in W_7^i}$ can be handled similarly with $k = 1$.

Since (20) is independent of $\xi$ and $m$, the proof is complete.

**References**


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