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SOME INFINITE SERIES IDENTITIES

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ABSTRACT. Certain infinite series are shown to satisfy simple identities between the square of the sum of the series and the sum of the squares of the terms of the series. The main tool is Ramanujan's ${}_1\psi_1$ summation formula.

It is unusual for an infinite series of nonzero terms to have the property that $\sum a_n^2 = (\sum a_n)^2$. In this note, we show how Ramanujan's ${}_1\psi_1$ summation can be used to derive a class of infinite series identities of this sort, the simplest of which is

$$(1) \qquad \sum_{k=-\infty}^{\infty} \left(\frac{(c/d)^k}{1-d^2c^{2k}} \right)^2 = \left(\sum_{k=-\infty}^{\infty} \frac{(c/d)^k}{1-d^2c^{2k}} \right)^2 \qquad \left(|c| < |d| < \left| \frac{1}{c} \right| \right).$$

Ramanujan's famous ${}_1\psi_1$ summation formula [3, Chapter 16, Entry 17] states that, for $|b/a| < |z| < 1$,

$$(2) \qquad \sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b)_k} z^k = \frac{(az)_\infty (q/az)_\infty (q)_\infty (b/a)_\infty}{(z)_\infty (b/az)_\infty (b)_\infty (q/a)_\infty},$$

where

$$(a)_\infty = (a\,;q)_\infty = \prod_{k=0}^{\infty} (1-aq^k) \qquad (|q|<1)$$

and

$$(a)_n = (a\,;q)_n = \frac{(a)_\infty}{(aq^n)_\infty} \qquad (n \in \mathbb{Z}).$$

A simple proof of (2) may be found in [2], and [1, pp. 30–34] contains a general discussion of the ${}_1\psi_1$ formula and some of its applications.

We first specialize the parameters of (2) by setting $q = c^2$, $a = d^2$, $b = d^2c^{2n}$, and $z = (c/d)e^{i\theta}$, where n is a positive integer and θ is real. Then

$$\frac{(a)_k}{(b)_k} = \frac{(d^2\,;c^2)_k}{(d^2c^{2n}\,;c^2)_k} = \frac{(d^2\,;c^2)_n}{(d^2c^{2k}\,;c^2)_n},$$

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and formula (2) becomes, after incorporating the term $(d^2; c^2)_n$ into the right-hand side,

$$\sum_{k=-\infty}^{\infty} \frac{(c/d)^k e^{ik\theta}}{(d^2c^{2k}; c^2)_n}$$
$$= \frac{(cde^{i\theta}; c^2)_\infty((c/d)e^{-i\theta}; c^2)_\infty(c^2; c^2)_\infty(c^{2n}; c^2)_\infty}{((c/d)e^{i\theta}; c^2)_\infty(dc^{2n-1}e^{-i\theta}; c^2)_\infty(d^2; c^2)_\infty(c^2/d^2; c^2)_\infty} =: f(\theta).$$

In particular, we have

$$f(0) = \sum_{k=-\infty}^{\infty} \frac{(c/d)^k}{(d^2c^{2k}; c^2)_n} = (cd; c^2)_{n-1}A_n, \tag{3}$$

where

$$A_n = \frac{(c^2; c^2)_\infty(c^{2n}; c^2)_\infty}{(d^2; c^2)_\infty(c/d^2; c^2)_\infty}.$$

Furthermore,

$$f(\theta)f(-\theta) = \sum_{k,l=-\infty}^{\infty} \frac{(c/d)^{k+l}e^{i\theta(k-l)}}{(d^2c^{2k}; c^2)_n(d^2c^{2l}; c^2)_n}$$
$$= A_n^2 \frac{(cde^{i\theta}; c^2)_\infty(cde^{-i\theta}; c^2)_\infty}{(dc^{2n-1}e^{i\theta}; c^2)_\infty(dc^{2n-1}e^{-i\theta}; c^2)_\infty}$$
$$= A_n^2(cde^{i\theta}; c^2)_{n-1}(cde^{-i\theta}; c^2)_{n-1}.$$

Integrating both sides with respect to θ over $(0, 2\pi)$ and dividing by 2π, we find that

$$\sum_{k=-\infty}^{\infty} \left(\frac{(c/d)^k}{(d^2c^{2k}; c^2)_n}\right)^2 = \frac{A_n^2}{2\pi}\int_0^{2\pi} (cde^{i\theta}; c^2)_{n-1}(cde^{-i\theta}; c^2)_{n-1}\, d\theta$$
$$= A_n^2 \sum_{m=0}^{n-1} a_{n,m}^2,$$

where we have written $(cde^{i\theta}; c^2)_{n-1} = a_{n,0} + a_{n,1}e^{i\theta} + \cdots + a_{n,n-1}e^{(n-1)i\theta}$. Combining this with (3), we deduce that

$$\left(\sum_{m=0}^{n-1} a_{n,m}\right)^2 \sum_{k=-\infty}^{\infty} \left(\frac{(c/d)^k}{(d^2c^{2k}; c^2)_n}\right)^2 = \sum_{m=0}^{n-1} a_{n,m}^2 \left(\sum_{k=-\infty}^{\infty} \frac{(c/d)^k}{(d^2c^{2k}; c^2)_n}\right)^2$$

provided $|c| < |d| < |c^{1-2n}|$. Identity (1) then follows by setting $n = 1$ and noting that $a_{1,0} = 1$.

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