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PRIMENESS OF TWISTED KNOTS

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ABSTRACT. Let V be a standardly embedded solid torus in S^3 with a meridian-preferred longitude pair (μ, λ) and K a knot contained in V. We assume that K is unknotted in S^3. Let f_n be an orientation-preserving homeomorphism of V which sends λ to $\lambda + n\mu$. Then we get a twisted knot $K_n = f_n(K)$ in S^3.

Primeness of twisted knots is discussed and we prove: A twisted knot K_n is prime if $|n| > 5$. Moreover, $\{K_n\}_{n\in Z}$ contains at most five composite knots.

1. INTRODUCTION

Let V be a standardly embedded solid torus in S^3 with a meridian-preferred longitude pair (μ, λ) and K a knot contained in V. We assume that K is unknotted in S^3, and K is not contained in a 3-ball in V. Let f_n be an orientation-preserving homeomorphism of V which sends λ to $\lambda + n\mu$. Then we obtain a twisted knot $K_n = f_n(K)$ in S^3. A simple question for this construction is: "Is a twisted knot K_n prime for any integer n?" [7]. (Here we think the unknot to be prime.) In connection with this problem we gave an example such that K_1 is a composite knot whose prime factors are torus knots of type (2, 3) and type (2, 5) (see Figure 1 on the next page) [10]. (Recently Ohyama also found such an example in which a trivial knot K creates a composite knot having the torus knot of type (2, 3) and the figure eight knot as prime factors by 1-twist.)

Herein we prove the following:

Theorem. *A twisted knot K_n is prime for any integer n satisfying $|n| > 5$. In addition, $\{K_n\}_{n\in Z}$ contains at most five composite knots.*

Remark. Recently Yasuhara [12] and Miyazaki [8] have shown independently an existence of composite knots which cannot be trivialized by any n-twist for $|n| \leq 5$, using a 4-dimensional technique. Hence these knots are able to be examples which cannot be obtained from trivial knot by twisting, and this answers the question proposed by Mathieu [7].

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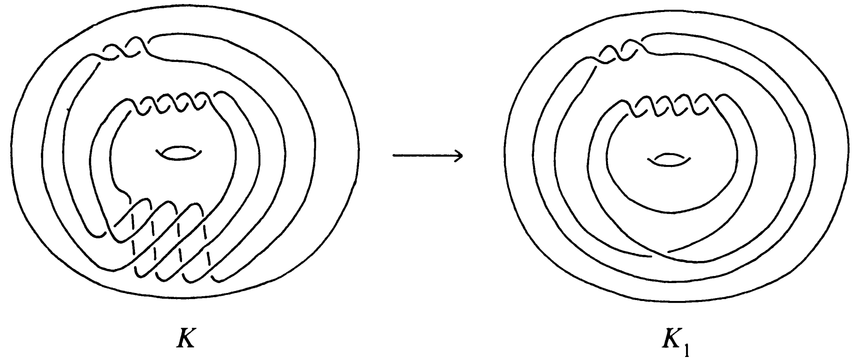

FIGURE 1

In the case where $\mathrm{wrap}_V(K)$—the minimal geometric intersection number of K and a meridian disk of V—is two, Scharlemann-Thompson [11], Gordon, and Zhang [13] have shown that K_n is always prime. (In [13], Zhang treated the case $n = \pm 1$, but the argument can easily be generalized to the case $|n| \geq 2$.)

The idea of the proof of the theorem depends upon that of Theorem 4.3 in [6]. But the recent developments, contributed by Gordon [2, 3], about Dehn fillings on hyperbolic manifolds enable us to sharpen the previous result.

Throughout this paper we use symbols ∂X, $\mathrm{int}\, X$, and $N(X)$ to denote the boundary of X, the interior of X, and the tubular neighborhood of X respectively.

2. Topological properties of an exterior $V - \mathrm{int}\, N(K)$

In this section, we prepare preliminary lemmas. We recali that, in our setting, a knot K is unknotted in S^3 and contained in a standardly embedded solid torus in S^3 and not contained in a 3-ball in V. Then $V - \mathrm{int}\, N(K)$ is an irreducible, boundary-irreducible Haken manifold, and we have a collection of tori $\mathcal{T}$ in $V - \mathrm{int}\, N(K)$ unique up to isotopy, which decomposes $V - \mathrm{int}\, N(K)$ so that each piece is simple or Seifert fibred [4, 5]. (Possibly $\mathcal{T} = \varnothing$.) Combining Thurston's uniformization theorem [9] and the torus theorem [4], a non-Seifert fibred piece admits a complete hyperbolic structure of finite volume in its interior. For simplicity, if the interior of X admits a complete hyperbolic structure of finite volume, then we say that X is hyperbolic. We denote the piece which contains ∂V (resp. $\partial N(K)$) by P_0 (resp. P_1). (Possibly $P_0 = P_1$.) From [4, Lemma VI.3.4], we see that each Seifert fibred piece in $V - \mathrm{int}\, N(K)$ is a torus knot space, a cable space, or a composing space.

The triviality of K in S^3 implies:

Lemma 2.1. *Let K be a knot in V which is unknotted in S^3. We assume that K is not a core of V and not contained in a 3-ball in V. Then we have:*

(1) *P_0 is not a composing space, and*
(2) *P_1 is not a composing space.*

Proof. First we consider the case $P_0 \neq P_1$. Suppose that P_0 is a composing space. Let T be a boundary component of P_0 which separates ∂V and $\partial N(K)$, and let W be the solid torus in V bounded by T. Let A be a saturated annulus (i.e., an annulus which is a union of fibres) in P_0 connecting ∂V

and T. We see that the boundary component of A contained in T bounds a meridian disk of W. Hence we have $\mathrm{wrap}_V(C_W) = 1$, where C_W is a core of W. By the assumption, ∂P_0 has at least three components. Let T' be a component of $\partial P_0 - (\partial V \cup T)$. Since T' bounds a nontrivial knot exterior in $V - \operatorname{int} N(K)$, it turns out that C_W has a locally knotted arc in V (i.e., there exists a 3-ball B in V such that $(B, B \cap C_W)$ is a knotted ball pair), and hence W is knotted in S^3. Moreover $\mathrm{wrap}_W(K) > 0$ holds. This implies that K is nontrivial in S^3 and contradicts the assumption.

If P_1 is a composing space, then we take a boundary component T'' of P_1 which separates ∂V and $\partial N(K)$. Let W' be the solid torus in V bounded by T''. Then we see that $\mathrm{wrap}_{W'}(K) = 1$ and K has a locally knotted arc in W' (i.e., there is a 3-ball B in W' such that $(B, B \cap K)$ is a knotted ball pair) by the same argument as above, and hence K is knotted in S^3. This contradicts the assumption again.

Next we consider the case $P_0 = P_1$. If P_0 ($= P_1$) is a composing space, then $\mathrm{wrap}_V(K) = 1$ and K has a locally knotted arc in V. Thus K is knotted in S^3. This is a contradiction and the proof is complete. □

3. Twistings, Dehn fillings, and torus decompositions

First notice that any simple loop γ on ∂V is parametrized by a meridian-preferred longitude pair (μ, λ) by which $\gamma = p\lambda + q\mu$ in $H_1(\partial V)$. We consider the manifold obtained by attaching the solid torus $S^1 \times D^2$ to $V - \operatorname{int} N(K)$ along ∂V so that afterwards γ bounds a disk in $S^1 \times D^2$. We say that the resulting manifold is obtained by $\frac{p}{q}$-*Dehn filling* on $V - \operatorname{int} N(K)$ along ∂V.

It is straightforward to see that the exterior $S^3 - \operatorname{int} N(K_n)$ is homeomorphic to the result of $-\frac{1}{n}$-Dehn filling on $V - \operatorname{int} N(K)$ along ∂V.

Composite knots can be characterized in terms of torus decompositions as follows [1, Lemma 4.1; 4, p. 183].

Lemma 3.1. *A knot k is a composite knot in S^3 if and only if the piece containing $\partial N(k)$ (with respect to a unique torus decomposition of $S^3 - \operatorname{int} N(k)$) is a composing space.*

In our situation, by Lemma 2.1(1), P_0 is a cable space or a hyperbolic piece. Let us denote the manifold obtained by $\frac{p}{q}$-Dehn filling on P_0 along ∂V by $P_0(\frac{p}{q})$.

To begin with, we treat the case where P_0 is a cable space.

Proposition 3.2. *Assume that P_0 is a cable space. Then K_n is prime for $n \neq 1$ or for $n \neq -1$.*

Proof. Suppose that a regular fibre t is presented by $p\lambda + q\mu$ ($p \geq 2$). Then the triviality of K in S^3 implies $q = \pm 1$. Thus a regular fibre of P_0 is presented by $p\lambda + \varepsilon\mu$ ($p \geq 2$ and $\varepsilon = \pm 1$) in our case. It follows that $P_0(-\frac{1}{n})$ is a Seifert fibred manifold with at most two exceptional fibres of indices p, $|pn + \varepsilon|$.

In case $\varepsilon = 1$, $|pn + \varepsilon| = 1$ only when $n = 0, -1$ and in case $\varepsilon = -1$, $|pn + \varepsilon| = 1$ only when $n = 0, 1$. Hence $P_0(-\frac{1}{n})$ is boundary-irreducible except for $n = 0, -1$ (resp. $0, 1$), if $\varepsilon = 1$ (resp. -1). From now on we assume that $n \neq 0, -1$ or $n \neq 0, 1$ according as $\varepsilon = 1$ or $\varepsilon = -1$. If

$P_0 \neq P_1$, it turns out that $\mathcal{J}$ gives a torus decomposition of $S^3 - \operatorname{int} N(K_n)$. Consequently P_1 is a decomposing piece in $S^3 - \operatorname{int} N(K_n)$, which is not a composing space by Lemma 2.1(2). It follows from Lemma 3.1 that K_n is a prime knot.

If $P_0 = P_1$, then $S^3 - \operatorname{int} N(K_n)$ is a torus knot exterior, and K_n is a prime knot. □

Next we consider the case where P_0 is hyperbolic. First we note the following.

Lemma 3.3. *Suppose that P_0 is hyperbolic. Then $P_0(\frac{1}{0})$ is not hyperbolic.*

Proof. If $P_0(\frac{1}{0})$ is hyperbolic, then $P_0(\frac{1}{0})$ is an irreducible and boundary-irreducible manifold. This implies that $S^3 - \operatorname{int} N(K)$ contains an incompressible torus ($\subset \partial P_0(\frac{1}{0})$). Hence K cannot be a trivial knot in S^3, and this is a contradiction. □

By making use of Gordon's recent result about Dehn fillings on hyperbolic manifolds [2, 3], we can obtain:

Lemma 3.4. *Suppose that P_0 is hyperbolic and that both $P_0(-\frac{1}{m})$ and $P_0(-\frac{1}{n})$ are nonhyperbolic. Then we have $|m - n| \leq 5$.*

Proof. Since ∂P_0 has at least two components, Gordon's result [2, 3] asserts that the distance of two slopes presented by $\lambda - m\mu$ and $\lambda - n\mu$ is less than or equal to five, that is, $|m - n| \leq 5$. This completes the proof. □

Proposition 3.5. *Assume that P_0 admits a complete hyperbolic structure of finite volume in its interior. Then K_n is prime when $|n| > 5$. Moreover, $\{K_n\}_{n \in Z}$ contains at most five composite knots.*

Proof. We note that if $P_0(-\frac{1}{n})$ is hyperbolic, then $\mathcal{J}$ gives a torus decomposition of $S^3 - \operatorname{int} N(K_n)$. (Possibly $\mathcal{J} = \varnothing$.) Therefore if $P_0 \neq P_1$, then P_1 is a decomposing piece in $S^3 - \operatorname{int} N(K_n)$ which contains $\partial N(K_n)$. By Lemma 2.1(2), P_1 is not a composing space, so we can conclude that K_n is a prime knot by Lemma 3.1. If $P_0 = P_1$, then $P_0(-\frac{1}{n})$ $(= P_1(-\frac{1}{n}))$ is a decomposing piece in $S^3 - \operatorname{int} N(K_n)$ which contains $\partial N(K_n)$. Since $P_0(-\frac{1}{n})$ is hyperbolic, it is not a composing space, and we see that K_n is a prime knot by Lemma 3.1.

By Lemma 3.3, $P_0(\frac{1}{0})$ is not hyperbolic, thus we see that $P_0(-\frac{1}{n})$ is hyperbolic if $|n| > 5$ by Lemma 3.4. In addition there are at most six integers n_i such that $P_0(-\frac{1}{n_i})$ is not hyperbolic by Lemma 3.4, and such a set $\{n_i\}$ contains zero and K_0 is a trivial knot. It follows that $\{K_n\}_{n \in Z}$ contains at most five composite knots. □

Now the proof of the Theorem is straightforward from Propositions 3.2 and 3.5.

We conclude this note with the following conjecture.

Conjecture. *A twisted knot K_n can be a composite knot only for one integer $n \in \{1, -1\}$.*

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