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AN EXAMPLE OF l_p-EQUIVALENT SPACES WHICH ARE NOT l_p^*-EQUIVALENT

JAN BAARS, JOOST DE GROOT, JAN VAN MILL, AND JAN PELANT

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ABSTRACT. We give an example of two locally compact countable metric spaces X and Y which are l_p-equivalent but not l_p^*-equivalent, i.e., $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic but $C_p^*(X)$ and $C_p^*(Y)$ are not linearly homeomorphic.

0. INTRODUCTION

Let X and Y be Tychonov spaces. By $C(X)$ (resp. $C^*(X)$), we denote the set of all real-valued continuous functions (resp. the set of all real-valued bounded continuous functions) on X. We endow $C(X)$ (resp. $C^*(X)$) with the topology of pointwise convergence and denote it by $C_p(X)$ (resp. $C_p^*(X)$). We define X and Y to be *l_p-equivalent* (resp. *l_p^*-equivalent*) whenever $C_p(X)$ and $C_p(Y)$ (resp. $C_p^*(X)$ and $C_p^*(Y)$) are linearly homeomorphic.

In [1] Baars and de Groot obtained a complete isomorphical classification for function spaces $C_p(X)$, where X is any locally compact zero-dimensional separable metric space. At this moment, an isomorphical classification for the corresponding function spaces $C_p^*(X)$ is not known. From the results in this paper it follows that such a classification must be different from the classification for $C_p(X)$. The main theorem in this paper states that for l_p^*-equivalent metric spaces X and Y we have that the scattered height of X is less than ω if and only if the scattered height of Y is less than ω. Together with the results in [1], this theorem gives us an example of two l_p-equivalent locally compact countable metric spaces which are not l_p^*-equivalent.

1. PRELIMINARIES

In this section we briefly discuss some standard terminology about derivatives of sets and some properties of function spaces which we need in the proofs of the results in §2.

Let X be a topological space, and let $A \subset X$. Recall that the derived set A^d of A in X is defined to be the set of all accumulation points of A in X. For

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every ordinal α we define $X^{(\alpha)}$, the *αth derivative*, by transfinite induction as follows:

(a) $X^{(0)} = X$;
(b) if α is a successor, say $\alpha = \beta + 1$, then $X^{(\alpha)} = (X^{(\beta)})^d$;
(c) if α is a limit ordinal then $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$.

Note that, for every ordinal α, $X^{(\alpha)}$ is closed in X, and $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$ (obviously, $(X^{(\alpha)})^{(1)}$ is the derived set of $X^{(\alpha)}$ in $X^{(\alpha)}$, whereas $X^{(\alpha+1)} = (X^{(\alpha)})^d$ is the derived set of $X^{(\alpha)}$ in X). If, moreover, $\beta \leq \alpha$ is an ordinal, then $X^{(\beta)} \subset X^{(\alpha)}$.

Let A be a subspace of X. A is *dense in itself* if $A \subset A^d$ or equivalently $A = A^{(1)}$. This means that A contains no isolated points. A is *scattered* if A contains no dense in itself subsets, i.e., every subset of A contains isolated points.

By the Cantor-Bendixson Theorem (cf. [8]), for any scattered space X, there is an ordinal α such that $X^{(\alpha)} = \varnothing$. The *scattered height* $\kappa(X)$ of a scattered space X is defined to be the smallest ordinal α such that $X^{(\alpha)} = \varnothing$. It is easy to see that if X is the ordinal space $\omega^\alpha + 1$, then $\kappa(X) = \alpha + 1$.

For a topological space X and a subset A of X, $C^*_{p,A}(X)$ denotes the subspace of $C^*_p(X)$ of all functions vanishing on A.

1.1. **Proposition.** *Let X be a metric space, and let A be a closed subset of X. Then $C^*_p(X) \sim C^*_{p,A}(X) \times C_p(A)$.*

Proof. Define $\rho\colon C^*_p(X) \to C^*_p(A)$ by $\rho(f) = f|A$. Then ρ is a continuous linear function. Because X is metric and A is closed, there is a continuous linear function $\xi\colon C^*_p(A) \to C^*_p(X)$ such that, for each $f \in C^*_p(A)$, $\xi(f)|A = f$ (cf. [4]).

Define $\phi\colon C^*_p(X) \to C^*_{p,A}(X) \times C^*_p(A)$ by $\phi(f) = (f - (\xi \circ \rho)(f), \rho(f))$. Then ϕ is a linear homeomorphism. □

We denote $C^*(X)$ with the topology of uniform convergence by $C^*_u(X)$. It is well known that $C^*_u(X)$ is a Banach space. Similar to $C^*_{p,A}(X)$, we define the subspace $C^*_{u,A}(X)$ of $C^*_u(X)$ to be the set of all elements of $C^*_u(X)$ which vanish on A. For $f \in C^*(X)$ and $\varepsilon > 0$, let $B(f, \varepsilon) = \{g \in C^*(X)\colon \sup\{|f(x) - g(x)|\colon x \in X\} < \varepsilon\}$.

In our proofs in §2 we need the Closed Graph Theorem, which states that, for Banach spaces E and F and a linear function $\phi\colon E \to F$ such that the set $\{(x, \phi(x))\colon x \in E\}$ is closed in $E \times F$, we have ϕ is continuous (cf. [6]). We use the Closed Graph Theorem, for example, in the following way: Let X and Y be spaces, and let $\phi\colon C^*_p(X) \to C^*_p(Y)$ be a continuous linear function. Then ϕ considered as a function from $C^*_u(X)$ to $C^*_u(Y)$ is also continuous.

2. The example

In this section we will prove, for l^*_p-equivalent metric spaces X and Y, that $\kappa(X) < \omega$ if and only if $\kappa(Y) < \omega$. The proof of this result is a generalization of Pelant's proof that $C^*_p(T)$ and $C^*_p(\mathbb{Q})$ are not linearly homeomorphic (cf. [7]). Here $\mathbb{Q}$ denotes the space of rationals and T the space $\mathbb{N}^2 \cup \{\infty\}$ where each point of $\mathbb{N}^2$ is isolated and $\{(\{n, n+1, \dots\} \times \mathbb{N}) \cup \{\infty\}\}_{n\in\mathbb{N}}$ is a local open base at ∞. The reader should compare this result with Theorem 2.11

in [2], which states that, for l_p-equivalent separable metric zero-dimensional spaces X and Y, $\kappa(X) \leq \omega$ if and only if $\kappa(Y) \leq \omega$ (in fact, this is even true for metric spaces; cf. [3]). Note that this theorem implies that $C_p(\mathbb{Q})$ and $C_p(T)$ are not linearly homeomorphic.

We first need the following definition, which can be found, for example, in [5]. A family $\mathscr{F} \subset C(X)$ is *equicontinuous* if, for every $x \in X$ and $\varepsilon > 0$, there is a neighborhood U of x in X such that, for each $f \in \mathscr{F}$ and $y \in U$, $|f(x) - f(y)| < \varepsilon$. The following result is well known. The proof is given for the sake of completeness.

2.1. **Proposition.** *If $\mathscr{F} \subset C_u^*(X)$ is compact, then $\mathscr{F}$ is equicontinuous.*

Proof. Let $x \in X$ and $\varepsilon > 0$. The family $\{B(f, \varepsilon/3): f \in \mathscr{F}\}$ is an open cover of $\mathscr{F}$. Since $\mathscr{F}$ is compact, there are $f_1, \dots, f_n \in \mathscr{F}$ $(n \in \mathbb{N})$ such that $\{B(f_i, \varepsilon/3): i \leq n\}$ covers $\mathscr{F}$. Since each f_i is continuous, there is a neighborhood U of x such that, for all $y \in U$ and for every $i \leq n$, $|f_i(y) - f_i(x)| < \varepsilon/3$. Now let $f \in \mathscr{F}$ and $y \in U$. There is $i \leq n$ such that $f \in B(f_i, \varepsilon/3)$. This implies $|f_i(x) - f(x)| < \varepsilon/3$ and $|f_i(y) - f(y)| < \varepsilon/3$. Since $y \in U$, we now have

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \varepsilon. \quad \square$$

2.2. **Theorem.** *Let X and Y be first countable l_p^*-equivalent spaces. Then $\kappa(X) < 2$ if and only if $\kappa(Y) < 2$.*

Proof. Suppose $\kappa(X) < 2$ and $\kappa(Y) \geq 2$. Since X cannot be empty, we have $\kappa(X) = 1$, which gives that X is discrete. Since $\kappa(Y) \geq 2$, there is $y \in Y$ which is nonisolated. Let $\{U_n: n \in \mathbb{N}\}$ be a decreasing open base at y in Y. For every $n \in \mathbb{N}$ let f_n be a Urysohn function with $f_n(y) = 1$ and $f_n(Y \setminus U_n) = 0$. Then $f_n \to \chi_{\{y\}}$ pointwise in $\mathbb{R}^Y$, where $\chi_{\{y\}}$ denotes the characteristic function of x. Since $\chi_{\{y\}} \notin C_p^*(Y)$, $\{f_n: n \in \mathbb{N}\}$ is closed and discrete in $C_p^*(Y)$.

Now let $\phi: C_p^*(X) \to C_p^*(Y)$ be a linear homeomorphism. Then by the Closed Graph Theorem, $\phi: C_u^*(X) \to C_u^*(Y)$ is also a linear homeomorphism. Since $C_u^*(X)$ and $C_u^*(Y)$ are Banach spaces, there is $k \in \mathbb{N}$ such that for every $f \in C^*(X)$ we have $\|f\|/k \leq \|\phi(f)\| \leq k\|f\|$. Let $g_n = \phi^{-1}(f_n)$. Then $\|g_n\| \leq k\|f_n\| = k$. Hence $\{g_n: n \in \mathbb{N}\} \subset [-k, k]^X$. Since $[-k, k]^X$ is compact, $\{g_n: n \in \mathbb{N}\}$ has an accumulation point $g \in [-k, k]^X$. Since X is discrete, $[-k, k]^X \subset C_p^*(X)$ and so $g \in C_p^*(X)$. However, since $\{f_n: n \in \mathbb{N}\}$ is closed and discrete in $C_p^*(Y)$, $\{g_n: n \in \mathbb{N}\}$ is closed and discrete in $C_p^*(X)$, which is a contradiction. $\square$

One could think for a moment that, for each $n \in \mathbb{N}$ and for all l_p^*-equivalent spaces X and Y, we have $\kappa(X) < n$ if and only if $\kappa(Y) < n$. For $n = 1$ it is trivially true, and for $n = 2$ it follows from Theorem 2.2. These are, however, the only cases in which it is true. If we take any $n \in \mathbb{N}$ with $n > 2$, we can find a counterexample. From Theorem 2.13 in [1] it follows that the ordinal spaces $\omega + 1$ and $\omega^{n-1} + 1$ are l_p^*-equivalent. Since ω^2 (resp. ω^n) is the topological sum of infinitely many copies of $\omega + 1$ (resp. $\omega^{n-1} + 1$), it follows that ω^2 and ω^n are l_p^*-equivalent. Note that the scattered height of ω^2 is 2 and the scattered height of ω^n is n.

Surprisingly enough the ordinal ω gives us again a positive answer (cf. Theorem 2.5). Before we prove this result we need two fairly simple lemmas. One deals with function spaces, and the other one deals with nets.

2.3. **Lemma.** *Let X be a metric space with $\kappa(X) < \omega$. There is a metric space Y such that $\kappa(Y) = \kappa(X)$ and $C_p^*(X) \sim C_{p,A}^*(Y)$ where $A = Y^{(1)}$.*

Proof. We prove the lemma by induction on $\kappa(X)$. If $\kappa(X) = 1$, let $Y = X$. So suppose the lemma has been proved for metric spaces X with $\kappa(X) < n$ $(n > 1)$. Let X be a metric space with $\kappa(X) = n$, and let $B = X^{(1)}$. Then, by Proposition 1.1, $C_p^*(X) \sim C_p^*(B) \times C_{p,B}^*(X)$. Since $\kappa(B) = n - 1$, there is by the inductive hypothesis a metric space Z such that $\kappa(Z) = \kappa(B)$ and $C_p^*(B) \sim C_{p,C}^*(Z)$ where $C = Z^{(1)}$. Then $C_p^*(X) \sim C_{p,C}^*(Z) \times C_{p,B}^*(X) = C_{p,B\cup C}^*(Z \oplus X)$ (the symbol "$\oplus$" stands for topological sum). Let $Y = Z \oplus X$. Then $Y^{(1)} = B \cup C$ and $\kappa(Y) = \kappa(X)$. This finishes the proof of the lemma. □

2.4. **Lemma.** *Let X be a space and B an infinite set. For every $b \in B$ let $f_b \in \mathbb{R}^X$ such that, for every $x \in X$, $\{b \in B: f_b(x) \neq 0\}$ is finite. Furthermore let $\mathscr{S} = \{S \subset B: S$ is finite$\}$ and define a relation $\leq$ on $\mathscr{S}$ as follows: If $S_1, S_2 \in \mathscr{S}$ then $S_1 \leq S_2$ if $S_1 \subset S_2$. For every $S \in \mathscr{S}$ define $f_S = \sum_{b\in S} f_b$. Then $\{f_S: S \in \mathscr{S}\}$ is a net in $\mathbb{R}^X$ and $\lim_{S\in\mathscr{S}} f_S = \sum_{b\in B} f_b$.*

Proof. It is easily seen that $\mathscr{S}$ is directed by $\leq$. Since every $S \in \mathscr{S}$ is finite, $f_S \in \mathbb{R}^X$; hence, $\{f_S: S \in \mathscr{S}\}$ is a net in $\mathbb{R}^X$.

Now let $\varepsilon > 0$ and $P \subset X$ be finite. For every $p \in P$ let $S_p = \{b \in B: f_b(p) \neq 0\}$ and $S_0 = \bigcup_{p\in P} S_p$. Then $S_0 \in \mathscr{S}$. Let $S \geq S_0$, $p \in P$, and $f = \sum_{b\in B} f_b$. Then

$$|f(p) - f_S(p)| = \left|\sum_{b\in B} f_b(p) - \sum_{b\in S} f_b(p)\right| = \left|\sum_{b\in S_p} f_b(p) - \sum_{b\in S_p} f_b(p)\right| = 0 < \varepsilon.$$

Hence, $\lim_{S\in\mathscr{S}} f_S = f$. □

We now come to the result announced in the introduction of this section.

2.5. **Theorem.** *Let X and Y be l_p^*-equivalent metric spaces. Then $\kappa(X) < \omega$ if and only if $\kappa(Y) < \omega$.*

Proof. Suppose $\kappa(X) < \omega$ and $\kappa(Y) \geq \omega$. By Lemma 2.3 we may assume $C_{p,A}^*(X) \sim C_p^*(Y)$ where $A = X^{(1)}$. Let $\phi: C_{p,A}^*(X) \to C_p^*(Y)$ be a linear homeomorphism. Then, by the Closed Graph Theorem, $\phi: C_{u,A}^*(X) \to C_u^*(Y)$ is also a linear homeomorphism. So there is $k \in \mathbb{N}$ such that for every $f \in C_{u,A}^*(X)$ we have $\|f\|/k \leq \|\phi(f)\| \leq k\|f\|$. Let $B = X\backslash A$. Since every element of B is an isolated point in X, we have for each $x \in B$ that $f_x = \chi_{\{x\}} \in C_{p,A}^*(X)$, where $\chi_{\{x\}}$ is the characteristic function of x. Notice that, for each $f \in C_{p,A}^*(X)$, $f = \sum_{x\in B} \alpha_x f_x$, where $\alpha_x = f(x)$. For each $x \in B$, let $g_x = \phi(f_x)$.

For every $y \in Y$, let $C_y = \{x \in B: g_x(y) \neq 0\}$.

Claim 1. C_y is finite for every $y \in Y$.

Suppose C_y is infinite for some $y \in Y$. Find an infinite subset $\{x_n: n \in \mathbb{N}\}$ in C_y. For $n \in \mathbb{N}$, define $h_n: X \to \mathbb{R}$ by $h_n = [1/g_{x_n}(y)] \cdot f_{x_n}$. Then $h_n \in$

$C^*_{p,A}(X)$ and $h_n \to 0$ $(n \to \infty)$ in $C^*_{p,A}(X)$. Now

$$\phi(h_n)(y) = [1/g_{x_n}(y)] \cdot \phi(f_{x_n})(y) = [1/g_{x_n}(y)] \cdot g_{x_n}(y) = 1.$$

Hence, $\phi(h_n) \nrightarrow 0$ $(n \to \infty)$ in $C^*_p(Y)$, which gives a contradiction, so Claim 1 is proved.

Now define $b\colon Y \to \mathbb{R}$ by $b(y) = \sum_{x \in B} |g_x(y)|$. Notice that, for every $y \in Y$, $b(y) = \sum_{x \in C_y} |g_x(y)|$; hence, b is well defined.

Claim 2. $\|b\| \le 2k$.

For $y \in Y$, let $C_y^+ = \{x \in B\colon g_x(y) > 0\}$ and $C_y^- = \{x \in B\colon g_x(y) < 0\}$. Notice that $\|\sum_{x \in C_y^+} g_x\| = \|\phi(\sum_{x \in C_y^+} f_x)\| \le k \cdot \|\sum_{x \in C_y^+} f_x\| = k$. Similarly we can prove that $\|\sum_{x \in C_y^-} g_x\| \le k$. So

$$|b(y)| = \left|\sum_{x \in C_y^+} g_x(y) - \sum_{x \in C_y^-} g_x(y)\right| \le \left|\sum_{x \in C_y^+} g_x(y)\right| + \left|\sum_{x \in C_y^-} g_x(y)\right| \le 2k,$$

which proves the claim.

Now for $P \subset B$ finite let $\mathscr{M}_P = \{\sum_{x \in P} \alpha_x f_x\colon |\alpha_x| \le k \text{ for } x \in P\}$. Notice that $\mathscr{M}_P = \prod_{x \in P}[-k, k] \times \prod_{x \in X \setminus P}\{0\}$.

Claim 3. For every $y \in Y$, $P \subset B$ finite, and $\varepsilon > 0$, there is a neighborhood $U(y, P, \varepsilon)$ of y in Y such that, for each $z \in U(y, P, \varepsilon)$ and $f \in \phi(\mathscr{M}_P)$, $|f(y) - f(z)| < \varepsilon$.

Notice that $\mathscr{M}_P$ is compact in $C^*_{p,A}(X)$. Since P is finite, it easily follows that $\mathscr{M}_P$ is compact in $C^*_{u,A}(X)$ and so $\phi(\mathscr{M}_P)$ is compact in $C^*_u(Y)$. Hence, by Proposition 2.1, $\phi(\mathscr{M}_P)$ is equicontinuous, from which the claim follows.

Now find $N \in \mathbb{N}$ such that $3(N+1)/4k \ge 2k$.

Claim 4. There are $y_0, \dots, y_N \in Y$, $P_0, \dots, P_N \subset B$ finite, and $U_0, \dots, U_N$ neighborhoods of $y_0, \dots, y_N$ respectively, such that

(1) for every $i \le N$: $C_{y_i} \subset P_i$,
(2) $P_0 \subset P_1 \subset \cdots \subset P_N$,
(3) $U_0 \supset U_1 \supset \cdots \supset U_N$,
(4) for every $i \le N$: $U_i \subset U(y_i, P_i, 1/4)$, and
(5) for every $i \le N$: $y_i \in Y^{(N-i)}$.

We will prove this claim by induction. Since $\kappa(Y) \ge \omega$, we can find $y_0 \in Y^{(N)}$. Let $P_0 = C_{y_0}$ and $U_0 = U(y_0, P_0, 1/4)$. Suppose $y_0, \dots, y_n$, $P_0, \dots, P_n$, and $U_0, \dots, U_n$ are found for $0 \le n < N$. Since $y_n \in Y^{(N-n)}$ and $N - n \ge 1$, we can find $y_{n+1} \in U_n \setminus \{y_i\colon i \le n\} \cap Y^{(N-(n+1))}$. Let $P_{n+1} = P_n \cup C_{y_{n+1}}$ and

$$U_{n+1} = U_n \cap U(y_{n+1}, P_{n+1}, \tfrac{1}{4}).$$

This completes the inductive construction and hence the proof of the claim.

Now let $g\colon Y \to [-1, 1]$ be a continuous function such that $g(y_i) = (-1)^i$ for $0 \le i \le N$. Then $\|g\| = 1$, so $\|\phi^{-1}(g)\| \le k$; hence, $\phi^{-1}(g) = \sum_{x \in B} \alpha_x f_x$ with $|\alpha_x| \le k$. Notice that $\sum_{x \in P_i} \alpha_x f_x \in \mathscr{M}_{P_i}$ for every $0 \le i \le N$.

Claim 5. $g = \sum_{x \in B} \alpha_x g_x$.

Indeed, let $\mathscr{S} = \{S \subset B : S \text{ is finite}\}$, and for every $S \in \mathscr{S}$ let $f_S = \sum_{x \in S} \alpha_x f_x$. By Lemma 2.4 $\phi^{-1}(g) = \lim_{S \in \mathscr{S}} f_S$ and $\sum_{x \in B} \alpha_x g_x = \lim_{S \in \mathscr{S}} \sum_{x \in S} \alpha_x g_x$. So

$$g = \phi(\phi^{-1}(g)) = \phi\left(\lim_{S \in \mathscr{S}} f_s\right) = \lim_{S \in \mathscr{S}} \phi(f_S) = \lim_{S \in \mathscr{S}} \sum_{x \in S} \alpha_x g_x = \sum_{x \in B} \alpha_x g_x,$$

and the claim is proved.

Let $0 \le i \le N$. Since $C_{y_i} \subset P_i$ (Claim 4(1)), we have, by Claim 5,

$$(-1)^i = \sum_{x \in B} \alpha_x g_x(y_i) = \sum_{x \in P_i} \alpha_x g_x(y_i).$$

By Claim 4(3) and (4), $y_N \in U(y_i, P_i, 1/4)$. Furthermore $\sum_{X \in P_i} \alpha_x g_x \in \phi(\mathscr{M}_{P_i})$, so, by Claim 3,

$$\left| \sum_{x \in P_i} \alpha_x g_x(y_N) - \sum_{x \in P_i} \alpha_x g_x(y_i) \right| < \frac{1}{4}.$$

If $i > 0$, we have by Claim 4(2)

$$\begin{aligned}
\left| \sum_{x \in P_i \setminus P_{i-1}} \alpha_x g_x(y_N) \right| &= \left| \sum_{x \in P_i} \alpha_x g_x(y_N) - \sum_{x \in P_{i-1}} \alpha_x g_x(y_N) \right| \\
&= \left| \sum_{x \in P_i} \alpha_x g_x(y_N) - (-1)^i + (-1)^{i-1} - \sum_{x \in P_{i-1}} \alpha_x g_x(y_N) \pm 2 \right| \\
&\ge 2 - \left| \sum_{x \in P_i} \alpha_x g_x(y_N) - \sum_{x \in P_i} \alpha_x g_x(y_i) \right| \\
&\quad - \left| \sum_{x \in P_{i-1}} \alpha_x g_x(y_N) - \sum_{x \in P_{i-1}} \alpha_x g_x(y_{i-1}) \right| > \frac{3}{4}.
\end{aligned}$$

If $i = 0$ and $P_{-1} = \varnothing$, then

$$\left| \sum_{x \in P_i \setminus P_{i-1}} \alpha_x g_x(y_N) \right| = \left| \sum_{x \in P_0} \alpha_x g_x(y_N) - \sum_{x \in P_0} \alpha_x g_x(y_0) + 1 \right| > 1 - \frac{1}{4} = \frac{3}{4}.$$

So by Claim 4(2)

$$\sum_{x \in P_N} |\alpha_x g_x(y_N)| \ge \sum_{i=0}^{N} \left| \sum_{x \in P_i \setminus P_{i-1}} \alpha_x g_x(y_N) \right| > \frac{3}{4}(N+1);$$

hence,

$$b(y_N) = \sum_{x \in P_N} |g_x(y_N)| \ge \sum_{x \in P_N} \left| \frac{\alpha_x}{k} g_x(y_N) \right| > \frac{3}{4k}(N+1) \ge 2k,$$

which gives a contradiction with Claim 2. □

2.6. **Example.** There are l_p-equivalent countable metric locally compact spaces which are not l_p^*-equivalent.

Proof. Let X and Y be the ordinal spaces ω^2 and ω^ω respectively. Then $\kappa(X) = 2$ and $\kappa(Y) = \omega$, so, by Theorem 2.5, X and Y are not l_p^*-equivalent spaces. However, by Theorems 2.13 and 3.14 in [1], X and Y are l_p-equivalent. □

In the proof of Theorem 2.5 the Closed Graph Theorem is applied to get a linear homeomorphism between the Banach spaces $C^*_{u,A}(X)$ and $C^*_u(Y)$. However, the proof also depends essentially on properties of the topology of pointwise convergence, so it does not give us a theorem for linear homeomorphisms between $C^*_u(X)$ and $C^*_u(Y)$. Consequently we cannot conclude that $C^*_u(\omega^2)$ and $C^*_u(\omega^\omega)$ are not linearly homeomorphic. It remains an open question whether $C^*_u(\omega^2)$ and $C^*_u(\omega^\omega)$ are linearly homeomorphic or not.

Recall that a prime component is an ordinal number of the form ω^μ for any ordinal μ. Motivated by Theorems 2.2 and 2.5 and the remark after Theorem 2.2 we state the following:

2.7. **Conjecture.** *Let X and Y be l_p^*-equivalent metric spaces, and let α be a prime component. Then*

(a) $\kappa(X) < \alpha$ *if and only if* $\kappa(Y) < \alpha$, *and*
(b) $\kappa(X) < \alpha + 1$ *if and only if* $\kappa(Y) < \alpha + 1$.

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(J. Baars) Faculteit Wiskunde en Informatica, Universiteit van Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

Current address: Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada M3J 1P3

E-mail address: janba@clid.yorku.ca

(J. de Groot, J. van Mill) Faculteit Wiskunde en Informatica, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

Current address, J. de Groot: Department of Mathematics and Computer Science, TU Delft, Postbus 5031, 2600 GA Delft, The Netherlands

E-mail address, J. de Groot: degroot@dutiaw3.tudelft.nl

E-mail address, J. van Mill: vanmill@cs.vu.nl

(J. Pelant) Matematický Ústav, Československá Akademie Věd, Žitná 25, 115 67 Praha, Československo

E-mail address: pelant@csearn.bitnet