

A REMARK ON WEIGHTED INEQUALITIES FOR GENERAL MAXIMAL OPERATORS

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ABSTRACT. Let $1 < p < \infty$, and let w, v be two nonnegative functions. We give a sufficient condition on w, v for which the general maximal operator $M_{\mathcal{Q}}$ is bounded from $L^p(v)$ into $L^p(w)$. Our condition is stronger but closely related to the $A_{p, \mathcal{Q}}$ condition for two weights.

1. INTRODUCTION

Let \mathcal{Q} be the family of all open cubes in \mathbf{R}^n with sides parallel to the axes, and let $M_{\mathcal{Q}} = M$ denote the Hardy-Littlewood maximal operator. According to a fundamental theorem of Sawyer [11], M is a bounded operator from $L^p(v)$ into $L^p(w)$, $1 < p < \infty$, if and only if $(w, v) \in S_p$, that is, there is a positive constant c such that

$$(1) \quad \int_Q M(v^{1-p'} \chi_Q)(y)^p w(y) dy \leq c \int_Q v(y)^{1-p'} dy, \quad Q \in \mathcal{Q}.$$

On the other hand, it is well known that Muckenhoupt's A_p condition for two weights,

$$(2) \quad \left(\frac{1}{|Q|} \int_Q w(y) dy \right)^{1/p} \left(\frac{1}{|Q|} \int_Q v(y)^{1-p'} dy \right)^{1/p'} \leq c, \quad Q \in \mathcal{Q},$$

is not equivalent to S_p unless $v = w$ (cf. [3; 2, p. 433]). One problem with Sawyer's condition is that it is very difficult to test in practice since it involves the operator M on it. It would be interesting to obtain sufficient conditions close in form to (2). In [10] we initiated this program and showed that it is enough to consider conditions such as (2) but replacing the local $L^{p'}$ average norm involved on the weight v by appropriate stronger norms. An antecedent of these results can be found in Neugebauer's paper [8].

This note is devoted to studying the corresponding two weight problem

$$(3) \quad \int_{\mathbf{R}^n} M_{\mathcal{B}} f(y)^p w(y) dy \leq c \int_{\mathbf{R}^n} |f(y)|^p v(y) dy,$$

where \mathcal{B} is a general basis. By a basis \mathcal{B} in \mathbf{R}^n , we mean a collection of open sets in \mathbf{R}^n . The study of general maximal operators $M_{\mathcal{B}}$ arises in many

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situations in Fourier Analysis where the geometry involved is other than the one given by the cubes or balls. Of course, in one-parameter Fourier Analysis the basis \mathcal{Q} plays a central role together with the basis \mathcal{D} of all dyadic cubes. A corresponding role in multiparameter Fourier Analysis is played by the basis \mathcal{R} of all rectangles with sides parallel to the axes. An example of an “exotic” but interesting basis is given by the Córdoba-Zygmund basis \mathfrak{R} in \mathbf{R}^3 of all rectangles with sidelengths of the form $\{s, t, st\}$.

For a general basis \mathcal{B} , $M_{\mathcal{B}}$ denotes the maximal operator associated to \mathcal{B} defined by

$$M_{\mathcal{B}}f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy$$

if $x \in \bigcup_{B \in \mathcal{B}} B$ and $M_{\mathcal{B}}f(x) = 0$, otherwise. We say that w is a weight associated to the basis \mathcal{B} if w is a nonnegative measurable function in \mathbf{R}^n such that $w(B) = \int_B w(y) dy < \infty$ for each B in \mathcal{B} . The weight w belongs to the class $A_{p, \mathcal{B}}$, $1 < p < \infty$, if there is a constant c such that

$$(4) \quad \left(\frac{1}{|B|} \int_B w(y) dy \right) \left(\frac{1}{|B|} \int_B w(y)^{1-p'} dy \right)^{p-1} \leq c$$

for all $B \in \mathcal{B}$. For $p = \infty$, we set $A_{\infty, \mathcal{B}} = \bigcup_{p > 1} A_{p, \mathcal{B}}$.

In [9] we introduced the following class of bases that we shall use later.

Definition 1.1. A basis \mathcal{B} is a *Muckenhoupt* basis if for each $1 < p < \infty$, and every $w \in A_{p, \mathcal{B}}$,

$$M_{\mathcal{B}}: L^p(w) \rightarrow L^p(w).$$

It is shown in [9] that this definition is equivalent to saying that for each $1 < p < \infty$

$$(5) \quad M_{\mathcal{B}, w}: L^p(w) \rightarrow L^p(w), \quad w \in A_{\infty, \mathcal{B}}.$$

Here $M_{\mathcal{B}, w}$ denotes the weighted maximal operator defined by

$$M_{\mathcal{B}, w}f(x) = \sup_{x \in B} \frac{1}{w(B)} \int_B |f(y)|w(y) dy.$$

Most of the important bases are Muckenhoupt bases, and, in particular, those mentioned above: \mathcal{Q} , \mathcal{D} , \mathcal{R} , and \mathfrak{R} .

Now let (w, v) be a couple of weights associated to the basis \mathcal{B} . Extending a previous result of Sawyer in [12] for the strong maximal operator, Jawerth [4] gave a necessary and sufficient condition for $M_{\mathcal{B}}$ to be bounded from $L^p(v)$ into $L^p(w)$ under no restriction on \mathcal{B} . Let \mathcal{F} be the family of all finite unions $G = \bigcup_{j=1}^N B_j$ of sets in \mathcal{B} ; Jawerth’s condition is that for some positive constant c

$$(6) \quad \int_G M_{\mathcal{B}}(v^{1-p'} \chi_G)(y)^p w(y) dy \leq c \int_G v(y)^{1-p'} dy, \quad G \in \mathcal{F}.$$

This condition is even harder to verify than Sawyer’s condition S_p . In Theorem 1.3 we shall provide a simpler sufficient condition that is not necessary.

We shall use the following class of weights.

Definition 1.2. We say that a weight w associated to the basis \mathcal{B} satisfies condition (A) if there are constants $0 < \lambda < 1$, $0 < c = c(\lambda) < \infty$ such that for all measurable sets E

$$(A) \quad w(\{x \in \mathbf{R}^n: M_{\mathcal{B}}(\chi_E)(x) > \lambda\}) \leq cw(E).$$

Before stating our main theorem we shall make some remarks concerning condition (A).

This class of weights was considered by Jawerth in [4], although the un-weighted version goes back to the work of Córdoba in [1]. One reason that makes condition (A) interesting is the fact that it is weaker than the $A_{\infty, \mathcal{B}}$ condition whenever the basis \mathcal{B} is a Muckenhoupt basis. To see this let $w \in A_{\infty, \mathcal{B}}$. Then $w \in A_{r, \mathcal{B}}$ for some $1 < r < \infty$ and, by standard properties of the $A_{p, \mathcal{B}}$ weights,

$$\frac{|E|}{|B|} \leq c \left(\frac{w(E)}{w(B)} \right)^{1/r},$$

for each measurable subset $E \subset B \in \mathcal{B}$. It follows then that $M_{\mathcal{B}}(\chi_E)(x) \leq c(M_{\mathcal{B}, w}(\chi_E)(x))^{1/r}$, and, therefore, if \mathcal{B} is a Muckenhoupt basis then (5) yields, for all $\lambda > 0$,

$$\begin{aligned} w(\{x \in \mathbf{R}^n: M_{\mathcal{B}}(\chi_E)(x) > \lambda\}) &\leq w \left(\left\{ x \in \mathbf{R}^n: M_{\mathcal{B}, w}(\chi_E)(x) > \frac{\lambda^r}{c^r} \right\} \right) \\ &\leq \frac{c^r}{\lambda^r} \int_{\mathbf{R}^n} \chi_E(x)^r w(x) dx = c(\lambda)w(E), \end{aligned}$$

which is condition (A).

In fact Jawerth and Torchinsky have shown in [5] that the $A_{\infty, \mathcal{B}}$ condition is strictly stronger than condition (A). As an example they show (cf. [5, p. 270]) that if the weight w in \mathbf{R}^n , $n > 1$, is A_{∞} in each variable except one where it is merely doubling, then w satisfies condition (A) while w does not belong to $A_{\infty, \mathcal{B}}$, as is well known.

A first result on the two weight problem was remarked by the author in [9]. We pointed out that the following generalized Fefferman-Stein's type inequality

$$(7) \quad \int_{\mathbf{R}^n} M_{\mathcal{B}} f(y)^p w(y) dy \leq c \int_{\mathbf{R}^n} |f(y)|^p M_{\mathcal{B}} w(y) dy$$

holds assuming that \mathcal{B} is a Muckenhoupt basis, and that the weight w satisfies condition (A). We recall that the classical Fefferman-Stein inequality for the Hardy-Littlewood maximal operator M has no restriction on w .

At this point we mention that a particular instance of this result was previously obtained by Lin in [6]. His result is for the strong maximal operator $M_{\mathcal{R}}$ in dimension $n = 2$ and with the weight w satisfying the $A_{\infty, \mathcal{R}}$ condition.

The main result of this paper is

Theorem 1.3. *Let $1 < p < \infty$, and let \mathcal{B} be a general basis satisfying $M_{\mathcal{B}}: L^s(\mathbf{R}^n) \rightarrow L^s(\mathbf{R}^n)$ for all $1 < s < \infty$. Suppose that (w, v) is a couple of weights such that w satisfies (A), and that for some $1 < r < \infty$ there is a constant c such that*

$$(8) \quad \frac{1}{|B|} \int_B w(y) dy \left(\frac{1}{|B|} \int_B v(y)^{(1-p')r} dy \right)^{(p-1)/r} \leq c,$$

for all $B \in \mathcal{B}$. Then

$$(9) \quad M_{\mathcal{B}}: L^p(v) \rightarrow L^p(w).$$

It is easy to check that $(w, M_{\mathcal{B}}w)$ satisfies (8), and then we have

Corollary 1.4. *Let $1 < p < \infty$, and let \mathcal{B} be a general basis satisfying $M_{\mathcal{B}}: L^s(\mathbf{R}^n) \rightarrow L^s(\mathbf{R}^n)$ for all $1 < s < \infty$. Suppose that w is a weight that satisfies condition (A). Then*

$$\int_{\mathbf{R}^n} M_{\mathcal{B}} f(y)^p w(y) dy \leq c \int_{\mathbf{R}^n} f(y)^p M_{\mathcal{B}} w(y) dy.$$

In particular, if \mathcal{B} is a Muckenhoupt basis then we always have that $M_{\mathcal{B}}: L^s(\mathbf{R}^n) \rightarrow L^s(\mathbf{R}^n)$, $1 < s < \infty$, yielding (7) as a corollary.

Corollary 1.5. *Let $1 < p < \infty$, and let \mathcal{B} be a Muckenhoupt basis. Suppose that w is a weight that satisfies condition (A). Then*

$$\int_{\mathbf{R}^n} M_{\mathcal{B}} f(y)^p w(y) dy \leq c \int_{\mathbf{R}^n} f(y)^p M_{\mathcal{B}} w(y) dy.$$

2. PROOF OF THE THEOREM

First we claim that (8) implies

$$(10) \quad M_{\mathcal{B}}: L^p(v) \rightarrow L^{p,\infty}(w),$$

where $L^{p,\infty}(w)$ is the weighted Lorentz space defined by all the functions f such that $\sup_{\lambda>0} (\lambda^p w(\{x \in \mathbf{R}^n: M_{\mathcal{B}} f(x) > \lambda\})) < \infty$.

To prove (10) we just need to prove that there is a constant c such that

$$w(K) \leq \frac{c}{t^p} \int_{\mathbf{R}^n} f(y)^p v(y) dy,$$

for each $t > 0$, $f \geq 0$, and for any compact subset K of $\{x \in \mathbf{R}^n: M_{\mathcal{B}}(f)(x) > t\}$. By the compactness of K and the definition of $M_{\mathcal{B}} f$, we can find a finite collection B_1, \dots, B_N such that

$$(11) \quad K \subset \bigcup_{j=1}^N B_j \quad \text{and} \quad t < \frac{1}{|B_j|} \int_{B_j} f(y) dy,$$

for each $j = 1, \dots, N$. We now follow a well-known selecting procedure argument (cf. [2, p. 463] for instance). Let $\tilde{B}_1 = B_1$ and, once $\tilde{B}_1, \dots, \tilde{B}_{k-1}$ have been selected, we choose \tilde{B}_k to be the first set in the given sequence (if any) such that

$$\left| \tilde{B}_k \cap \left(\bigcup_{j=1}^{k-1} \tilde{B}_j \right) \right| < \lambda |\tilde{B}_k|.$$

Now we claim that

$$(12) \quad \bigcup_{j=1}^N B_j \subset \{x \in \mathbf{R}^n: M_{\mathcal{B}}(\chi_{\bigcup_{j=1}^M \tilde{B}_j})(x) \geq \lambda\}.$$

Let $x \in \bigcup_{j=1}^N B_j$; if x belongs to some \tilde{B}_k it is of course obvious that it is contained on the set to the right since $\lambda < 1$. If, on the other hand, $x \in B_j$

for some B_j that has been discarded in the selection process, we must have $|B_j \cap (\bigcup_{j=1}^M \tilde{B}_j)| \geq \lambda|B_j|$, and therefore $M_{\mathcal{B}}(\chi_{\bigcup_{j=1}^M \tilde{B}_j})(x) \geq \lambda$. Now, since w satisfies condition (A), (12) yields

$$w\left(\bigcup_{j=1}^N B_j\right) \leq cw\left(\bigcup_{j=1}^M \tilde{B}_j\right).$$

This together with (11), (8), and Hölder’s inequality yields the estimate

$$\begin{aligned} w(K) &\leq cw\left(\bigcup_{j=1}^M \tilde{B}_j\right) \leq c \sum_j \left(\frac{1}{t|\tilde{B}_j|} \int_{\tilde{B}_j} f(y) dy\right)^p w(\tilde{B}_j) \\ &= c \frac{1}{t^p} \sum_j \left(\frac{1}{|\tilde{B}_j|} \int_{\tilde{B}_j} f(y)v(y)^{1/p}v(y)^{-1/p} dy\right)^p w(\tilde{B}_j) \\ &\leq c \frac{1}{t^p} \sum_j \left(\frac{1}{|\tilde{B}_j|} \int_{\tilde{B}_j} f(y)^{(p'r)'}v(y)^{(p'r)'/p} dy\right)^{p/(p'r)'} \\ &\quad \times \left(\frac{1}{|\tilde{B}_j|} \int_{\tilde{B}_j} v(y)^{-p'r/p} dy\right)^{p/p'r} \frac{w(\tilde{B}_j)}{|\tilde{B}_j|} \\ &\leq cK \frac{1}{t^p} \sum_j \left(\frac{1}{|\tilde{B}_j|} \int_{\tilde{B}_j} f(y)^{(p'r)'}v(y)^{(p'r)'/p} dy\right)^{p/(p'r)'} |\tilde{B}_j|. \end{aligned}$$

Denote $E_j = \tilde{B}_j \setminus \bigcup_{i=1}^{j-1} \tilde{B}_i$, so that $\{E_j\}$ is a disjoint family with $E_j \subset \tilde{B}_j$ and $|\tilde{B}_j| < \frac{1}{1-\lambda}|E_j|$. Then

$$\begin{aligned} w(K) &\leq c \frac{1}{t^p} \sum_j \left(\frac{1}{|\tilde{B}_j|} \int_{\tilde{B}_j} f(y)^{(p'r)'}v(y)^{(p'r)'/p} dy\right)^{p/(p'r)'} |E_j| \\ &\leq \frac{c}{t^p} \sum_j \int_{E_j} M_{\mathcal{B}}(f^{(p'r)'}v^{(p'r)'/p})(y)^{p/(p'r)'} dy \\ &\leq \frac{c}{t^p} \int_{\mathbf{R}^n} M_{\mathcal{B}}(f^{(p'r)'}v^{(p'r)'/p})(y)^{p/(p'r)'} dy \\ &\leq \frac{c}{t^p} \int_{\mathbf{R}^n} f(y)^p v(y) dy, \end{aligned}$$

since, by hypothesis, $M_{\mathcal{B}}: L^s(\mathbf{R}^n) \rightarrow L^s(\mathbf{R}^n)$, $1 < s < \infty$.

To conclude the proof of the theorem we observe first that we always have

$$(13) \quad M_{\mathcal{B}}: L^\infty(v) \rightarrow L^\infty(w).$$

Now, denoting condition (8) by $A_{p,r}$, we see that (w, v) satisfies $A_{\bar{p},\bar{r}}$ for some $1 < \bar{p} < p$, $1 < \bar{r} < \infty$; in fact $(p-1)/r + 1 < \bar{p} < p$, $1 < \bar{r} <$

$(\bar{p} - 1)r/(p - 1)$ will do it:

$$\begin{aligned} & \frac{1}{|B|} \int_B w(y) dy \left(\frac{1}{|B|} \int_B (v(y)^{-1})^{\bar{r}/(\bar{p}-1)} dy \right)^{(\bar{r}/(\bar{p}-1))^{-1}} \\ & \leq \frac{1}{|B|} \int_B w(y) dy \left(\frac{1}{|B|} \int_B (v(y)^{-1})^{r/(p-1)} dy \right)^{(r/(p-1))^{-1}} \leq K. \end{aligned}$$

By the above argument $A_{\bar{p}, \bar{r}}$ and $M_{\mathcal{B}}: L^s(\mathbf{R}^n) \rightarrow L^s(\mathbf{R}^n)$, $1 < s < \infty$, yield

$$M_{\mathcal{B}}: L^{\bar{p}}(v) \rightarrow L^{\bar{p}, \infty}(w).$$

This together with (13) implies $M_{\mathcal{B}}: L^p(v) \rightarrow L^p(w)$ by the Marcinkiewicz interpolation theorem. This concludes the proof of the theorem. \square

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