DERIVATIONS WITH INVERTIBLE VALUES
ON A MULTILINEAR POLYNOMIAL

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Abstract. Let $R$ be a semiprime $K$-algebra with unity, $d$ a nonzero derivation of $R$, and $f(x_1, \ldots, x_t)$ a monic multilinear polynomial over $K$ such that $d(f(a_1, \ldots, a_t)) \neq 0$ for some $a_1, \ldots, a_t \in R$. It is shown that if for every $r_1, \ldots, r_t$ in $R$ either $d(f(r_1, \ldots, r_t)) = 0$ or $d(f(r_1, \ldots, r_t))$ is invertible in $R$, then $R$ is either a division ring $D$ or $M_2(D)$, the ring of $2 \times 2$ matrices over $D$, unless $f(x_1, \ldots, x_t)$ is a central polynomial for $R$.

Moreover, if $R = M_2(D)$, where $2R \neq 0$ and $f(x_1, \ldots, x_t)$ is not a central polynomial for $D$, then $d$ is an inner derivation of $R$.

In [4] Bergen, Herstein, and Lanski proved that if $R$ is a ring with unity and $d \neq 0$ is a derivation of $R$ such that for every $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in $R$, then except for a special case which occurs when $2R = 0$, $R$ must be a division ring $D$ or $M_2(D)$, the ring of $2 \times 2$ matrices over a division ring $D$. In [5] Bergen and Carini gave a generalization of this result to the case of a Lie ideal. More precisely, for the semiprime case they proved: Let $R$ be a semiprime ring with $1$, $U$ a noncentral Lie ideal of $R$ such that $d(U) \neq 0$, and $d(u) = 0$ or $d(u)$ is invertible for every $u \in U$. Then $R$ is either $D$ or $M_2(D)$ for some division ring $D$. Moreover, if $R = M_2(D)$, where $D$ is not commutative and $2R \neq 0$, then $d$ must be inner.

Since by [9, Theorem 1.5] every noncentral Lie ideal of a simple ring $R$ must contain all commutators $xy - yx$ with $x, y \in R$ except if $R$ is of characteristic 2 and is 4-dimensional over its center, it is natural to examine what happens when the Lie ideal in Bergen and Carini's theorem is replaced by a multilinear polynomial.

Throughout this paper $R$ always denotes a semiprime $K$-algebra with unity where $K$ is a commutative ring with 1. A polynomial $f(x_1, \ldots, x_t)$ in $K\{x_1, x_2, \ldots\}$, the free $K$-algebra with indeterminates $x_i$, is called monic if $f(x_1, \ldots, x_t)$ contains some monomial with coefficient 1. In this paper we shall prove the following

Main Theorem. Let $R$ be a semiprime $K$-algebra with unity, $d$ a nonzero derivation of $R$, and $f(x_1, \ldots, x_t)$ a monic multilinear polynomial over $K$ such that $d(f(a_1, \ldots, a_t)) \neq 0$ for some $a_i \in R$. Suppose that for every

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\(r_1, \ldots, r_t\) in \(R\) either \(d(f(r_1, \ldots, r_t)) = 0\) or \(d(f(r_1, \ldots, r_t))\) is invertible in \(R\). Then \(R\) is either a division ring \(D\) or \(M_2(D)\), the ring of \(2 \times 2\) matrices over \(D\), unless \(f(x_1, \ldots, x_t)\) is a central polynomial for \(R\). Moreover, if \(R = M_2(D)\), where \(2R \neq 0\) and \(f(x_1, \ldots, x_t)\) is not a central polynomial for \(D\), then \(d\) must be an inner derivation of \(R\).

Given two elements \(a, b \in R\), \([a, b]\) will denote the element \(ab - ba\); also for two subsets \(A, B\) of \(R\), \([A, B]\) is then the additive subgroup of \(R\) generated by all \([a, b]\) for \(a \in A\) and \(b \in B\). \(Z(R)\) (or \(Z\) in brief) stands for the center of \(R\). We also recall that a polynomial \(f(x_1, \ldots, x_t) \in K\{x_1, x_2, \ldots\}\) is called a central polynomial for \(R\) if \(f(r_1, \ldots, r_t) \in Z(R)\) for all \(r_1, \ldots, r_t \in R\). For any subset \(S\) of \(R\), denote by \(l_R(S)\) the left annihilator of \(S\) in \(R\); that is, \(l_R(S) = \{x \in R \mid xS = 0\}\). We define \(r_R(S)\) similarly.

We begin this paper with the following

**Theorem 1.** Let \(R\) be a semiprime \(K\)-algebra with unity, \(d\) a nonzero derivation of \(R\), and \(f(x_1, \ldots, x_t)\) a monic polynomial without constant term, not necessarily multilinear, over \(K\) such that \(d(f(a_1, \ldots, a_t)) \neq 0\) for some \(a_i \in R\) and \(f(x_1, \ldots, x_t)\) is not a central polynomial for \(R\). Suppose that for every \(r_1, \ldots, r_t \in R\) either \(d(f(r_1, \ldots, r_t)) = 0\) or \(d(f(r_1, \ldots, r_t))\) is invertible in \(R\). Then \(R\) is either

(I) a division ring \(D\),

(II) \(M_2(D)\), the ring of \(2 \times 2\) matrices over a division ring \(D\), or

(III) \(M_n(\Delta)\) for some finite-dimensional central division algebra \(\Delta\) and some positive integer \(n\).

Moreover, if \(f(b_1, \ldots, b_t)\) is an element in \(R\) of rank \(m \geq 1\) for some \(b_i \in R\), then \(R\) assumes (III) only if \(n < 2m\).

**Proof.** We first claim that \(R\) is a simple ring with 1. Indeed, let \(I\) be a proper ideal of \(R\) and \(y_1, \ldots, y_t \in I^2\). Then it is clear that \(d(f(y_1, \ldots, y_t)) \in I\). Since either \(d(f(y_1, \ldots, y_t)) = 0\) or \(d(f(y_1, \ldots, y_t))\) is invertible, we have \(d(f(y_1, \ldots, y_t)) = 0\) for all \(y_i \in I^2\). By [12, Theorem 4] we have \(d(f(x_1, \ldots, x_t)y) = 0\) for all \(x_i \in R\) and all \(y \in I^2\). But by hypothesis \(d(f(a_1, \ldots, a_t))\) is invertible; this implies \(I^2 = 0\) and hence \(I = 0\) by the semiprimeness of \(R\). Thus \(R\) is a simple ring. We divide the proof into two cases.

**Case 1.** Assume that there exists a nonzero right ideal \(\rho\) of \(R\) such that \(d(f(x_1, \ldots, x_t)) = 0\) for all \(x_i \in \rho\).

Denote by \(g(x_1, \ldots, x_m)\) the multilinearization of \(f(x_1, \ldots, x_t)\). Since \(f \neq 0\) we have \(g \neq 0\). By assumption \(d(g(x_1, \ldots, x_m)) = 0\) is an identity for \(\rho\). Thus for any \(u \in \rho\) we have

\[
\sum_{i=1}^{m} g(u, x_i) + d(u)x_i + d(u)x_i, \ldots, u, x_m) = 0
\]

for all \(x_i \in R\). If \(d\) is an outer derivation of \(R\), by Kharchenko's theorem [11] (1) is reduced to \(\sum_{i=1}^{m} g(u, x_i) + d(u)x_i, \ldots, u, x_m) = 0\) for all \(x_i, y_j \in R\). In particular, \(g(u, x_i, \ldots, u, x_m) = 0\) for all \(x_i \in R\). If \(\rho \subseteq Z(R)\), then \(R\) is just a field and hence we are done. So we may assume \(\rho \not\subseteq Z(R)\). Choose an element \(u \in \rho \setminus Z(R)\). Then \(g(u, x_i, \ldots, u, x_m) = 0\) is a nontrivial generalized polynomial identity for \(R\). Thus by Martindale's theorem [14] \(R\)
is a strongly primitive ring. Since $R$ is a simple ring with $1$, $R$ is a finite-dimensional central simple algebra. That is, $R$ assumes the form (III). Thus we may assume that $d = ad(b)$, the inner derivation induced by some $b \in R$. That is, $d(x) = bx - xb$ for all $x \in R$. In this case, we have

$$[b, g(x_1, \ldots, x_m)] = 0 \quad \text{for all } x_i \in \rho.$$  

Assume first that $(b - \alpha)p = 0$ for some $\alpha \in Z(R)$. Choosing $u \in p \setminus Z(R)$ and using (2) we obtain that $g(ux_1, \ldots, ux_m)(b - \alpha) = 0$ for all $x_i \in R$. Thus $R$ assumes the form (III) as before. So we assume that $(b - \alpha)p \neq 0$ for any $\alpha \in Z(R)$. Then there exists an element $u \in p$ such that $bu$ and $u$ are linearly independent over $Z(R)$. Now by (2) we yield that $bg(ux_1, \ldots, ux_m) - g(ux_1, \ldots, ux_m)b = 0$ is a nontrivial generalized polynomial identity for $R$. As before, $R$ assumes the form (III).

Case 2. Assume that $d(f(\rho)) \neq 0$ for all nonzero right ideals $\rho$ of $R$.

The proof of this case is essentially that of [4, Lemma 4]. For any nonzero right ideal $\rho$ of $R$ we have $0 \neq d(f(\rho)) \subseteq d(\rho) + \rho \subseteq d(\rho) + p$ since $f$ has no constant term and $d(\rho) + \rho$ is a right ideal of $R$. But $0 \neq d(f(\rho))$ contains invertible values; this implies $d(\rho) + p = R$. Let $\rho_1, \rho_2$ be right ideals of $R$ such that $0 \neq \rho_1 \subset \rho_2$. We want to prove that $\rho_2 = R$. Indeed, this will imply that $R$ always assumes either (I) or (II). Note that $d(\rho_1) + \rho_1 = R = d(\rho_2) + \rho_2$. Choose an element $t \in \rho_1 \setminus \rho_1$. Write $t = a + d(b)$ for some $a, b \in \rho_1$. Then $d(b) \neq 0$ and $d(b) = t - a \in \rho_2$. Since $bR$ is a nonzero right ideal of $R$, we have $bR + d(bR) = R$. But $d(bR) \subseteq bd(R) + d(b)R \subseteq \rho_2$, thus we have $R = bR + d(bR) \subseteq \rho_2$, and hence $R = \rho_2$ as desired. This completes the first part of the theorem.

Finally, suppose that $R = M_n(A)$ as given in (III) and that rank$(f(b_1, \ldots, b_l)) = m \geq 1$ for some $b_i \in R$. We want to prove $n \leq 2m$. Assume on the contrary that $n > 2m$. Since rank$(f(b_1, \ldots, b_l)) = m$, $f(b_1, \ldots, b_l) = gx$ for some $x \in R$ and some idempotent $g \in R$ with rank $m$. Thus $d(f(b_1, \ldots, b_l)) = d(gx) = gd(gx) + d(g)gx$ and hence rank$(d(f(b_1, \ldots, b_l))) \leq 2m$. So $d(f(b_1, \ldots, b_l)) = 0$. Now consider the additive subgroup $A$ of $R$ generated by elements of rank $m$ assuming the form $f(u_1, \ldots, u_l)$ for some $u_i \in R$. Since $f(b_1, \ldots, b_l) \in A$, $A$ is a noncentral additive subgroup of $R$. Clearly, $A$ is invariant under special automorphisms in the sense of [7]. Thus by [7, Theorem 1] $A$ contains a noncentral Lie ideal of $R$, i.e., $A \supseteq [R, R]$. But $d(A) = 0$, thus we get $d([R, R]) = 0$. Also, dim$_R R > 4m^2$, which implies $d = 0$, a contradiction. So $n \leq 2m$ as desired. This completes the proof.

To prove the Main Theorem we need some notation from [13]. Let $S$ be a ring with $1$ and let $e_{ij}$ be the usual matrix units in the $n \times n$ matrix ring $M_n(S)$. Recall that for a sequence $u = (A_1, \ldots, A_k)$ in $M_n(S)$ the value of $u$ is defined to be the product $|u| = A_1A_2 \cdots A_k$ and $u$ is nonvanishing if $|u| \neq 0$. For a permutation $\sigma$ of $\{1, 2, \ldots, k\}$ we write $u^\sigma = (A_{\sigma(1)}, \ldots, A_{\sigma(k)})$. We call $u$ simple if it has the form $u = (a_1e_{ij}, \ldots, a_ke_{ij})$, where $a_i \in S$, $i = 1, \ldots, k$. Finally, a simple sequence $u$ is called even if for some $\sigma$, $|u^\sigma| = be_{ii} \neq 0$, and odd if for some $\sigma$, $|u| = be_{ij} \neq 0$, where $i \neq j$.

Before giving the proof of the Main Theorem we need the following result which is interesting in itself. The proof of the following lemma is implicit in [13, Lemma 2, proof of Lemma 3].
Lemma. Let $S$ be a $K$-algebra with 1 and let $R = M_n(S)$, $n \geq 2$. Suppose that $h(x_1, \ldots, x_t)$ is a multilinear polynomial over $K$ such that $h(u) = 0$ for all odd simple sequences $u$. Then $h(x_1, \ldots, x_t)$ is a central polynomial for $R$.

Proof. Let $B_1, \ldots, B_t \in R$ be arbitrary. Since $h(x_1, \ldots, x_t)$ is multilinear, we have $h(B_1, \ldots, B_t) = \sum_{l=1}^m h(u(l))$, where the $u(l)$ are even simple sequences because $h(u) = 0$ for any odd simple sequence $u$. By [13, Lemma 2] each $h(u(l))$ assumes a diagonal form. Thus $h(B_1, \ldots, B_t)$ always assumes a diagonal form. Write

$$h(B_1, \ldots, B_t) = \sum_{j=1}^n \beta_j e_{jj},$$

where $\beta_j \in S$. For $1 < k \leq n$ and any $\delta \in S$ we have that

$$h((1 + \delta e_{1k})B_1(1 + \delta e_{1k})^{-1}, \ldots, (1 + \delta e_{1k})B_t(1 + \delta e_{1k})^{-1})$$

still assumes a diagonal form. However,

$$h((1 + \delta e_{1k})B_1(1 + \delta e_{1k})^{-1}, \ldots, (1 + \delta e_{1k})B_t(1 + \delta e_{1k})^{-1})$$

$$= (1 + \delta e_{1k})h(B_1, \ldots, B_t)(1 + \delta e_{1k})^{-1}$$

$$= \sum_{j=1}^n \beta_j e_{jj} + (\delta \beta_k - \beta_1 \delta)e_{1k}, \quad \text{since } (1 + \delta e_{1k})^{-1} = 1 - \delta e_{1k}.$$

Thus $\delta \beta_k = \beta_1 \delta$ for all $\delta \in S$. In particular, set $\delta = 1$; then $\beta_1 = \beta_k$. So $\delta \beta_1 = \beta_1 \delta$ for all $\delta \in S$, which implies $\beta_1 \in Z(S)$. Now $h(B_1, \ldots, B_t) = \beta_1 \cdot \sum_{j=1}^n e_{jj} \in Z(R)$ as desired. This completes the proof.

Proof of Main Theorem. Assume that $f(x_1, \ldots, x_t)$ is not a central polynomial for $R$. By Theorem 1, $R = M_n(\Delta)$ for some division ring $\Delta$, and to prove $n < 2$ it suffices to show that rank$(f(b_1, \ldots, b_t)) = 1$ for some $b_i \in R$. By the previous lemma there exists an odd simple sequence $u$ such that $f(u) \neq 0$. But by [13, Lemma 2] $f(u) = \mu e_{ij} \neq 0$ for some $\mu \in \Delta$, $i \neq j$; we get rank$(f(u)) = 1$ as claimed.

Suppose next that $R = M_2(D)$, where $2R \neq 0$ and $f(x_1, \ldots, x_t)$ is not a central polynomial for the division ring $D$. We want to prove that $d$ is inner. To do this we will refer to some arguments given in [4, Lemma 8; 5, Lemma 10]. Since $d$ is a derivation of $R$, $d$ has the form:

$$d \begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} \phi(a) - b\beta - \alpha c & \phi(b) + a\alpha + b\gamma - \alpha e \\ \phi(c) + \beta a - e\beta - \gamma c & \phi(e) + e\gamma - \gamma e + \beta b + c\alpha \end{pmatrix}$$

for all $a, b, c, e \in D$, where $\alpha, \beta, \gamma \in D$ and $\phi$ is a derivation of $D$. Furthermore, by [4, Lemma 7] $d$ is inner on $M_2(D)$ if and only if $\phi$ is inner on $D$. Thus the aim is to prove that $\phi$ is inner on $D$. Suppose that $\alpha = 0$. Then for $\beta_1, \ldots, \beta_t \in D$ we have

$$d \begin{pmatrix} f(\beta_1, \ldots, \beta_t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} f(\beta_1, \ldots, \beta_t) \\ 0 \\ 0 \end{pmatrix},$$

which is zero or invertible. Thus $\phi(f(\beta_1, \ldots, \beta_t)) = 0$ for all $\beta_j \in D$. Let $T$ denote the subdivision ring of $D$ generated by all elements $f(\beta_1, \ldots, \beta_t)$,
where $\beta_i \in D$. Thus $\phi(T) = 0$. Since $f$ is noncentral on $D$, $T$ is then a noncentral subdivision ring of $D$ invariant under all automorphisms. By a result of Brauer-Cartan-Hua [6], $T = D$ follows. Thus $\phi(D) = 0$, implying that $\phi$ is inner. So we assume from now on that $\alpha \neq 0$. By (1) we have for $a \in D$ that
\[
d\left(\begin{array}{cc}
a & 0 \\
\alpha^{-1}\phi(a) & \alpha^{-1}a\alpha
\end{array}\right) = \left(\begin{array}{c}
0 \\
u
\end{array}\right),
\]
where
\[
u = \phi(\alpha^{-1}\phi(a)) + \beta a - \alpha^{-1}a\beta - \gamma a^{-1}\phi(a),
\]
\[
\begin{align*}
v &= \phi(\alpha^{-1}a\alpha) + \alpha^{-1}a\alpha\gamma - \gamma a^{-1}a\alpha + \alpha^{-1}\phi(a)\alpha.
\end{align*}
\]
Note that for $a, b \in D$ we have
\[
\begin{align*}
&\begin{pmatrix}
a & 0 \\
\alpha^{-1}\phi(a) & \alpha^{-1}a\alpha
\end{pmatrix}
\begin{pmatrix}
b & 0 \\
\alpha^{-1}\phi(b) & \alpha^{-1}b\alpha
\end{pmatrix}
= \begin{pmatrix}
ab & 0 \\
\alpha^{-1}\phi(ab) & \alpha^{-1}ab\alpha
\end{pmatrix}.
\end{align*}
\]
Thus for $\beta_1, \ldots, \beta_t \in D$ we have
\[
d\left(\begin{array}{c}
f(\beta_1, \ldots, \beta_t) \\
\alpha^{-1}\phi(f(\beta_1, \ldots, \beta_t))
\end{array}\right)
\begin{align*}
&= d\left(f\left(\begin{pmatrix}
\beta_1 & 0 \\
\alpha^{-1}\phi(\beta_1) & \alpha^{-1}\beta_1\alpha
\end{pmatrix}, \ldots, \begin{pmatrix}
\beta_t & 0 \\
\alpha^{-1}\phi(\beta_t) & \alpha^{-1}\beta_t\alpha
\end{pmatrix}\right)\right),
\end{align*}
\]
which is either 0 or invertible. By (2) it must be zero. Now using the same calculations given in [4, Lemma 8] we have
\[
\phi(f(\beta_1, \ldots, \beta_t)) = \frac{1}{2}[\phi(\alpha)\alpha^{-1} + \alpha\gamma\alpha^{-1}, f(\beta_1, \ldots, \beta_t)]
\]
for all $\beta_1, \ldots, \beta_t \in D$. Assume on the contrary that $\phi$ is outer on $D$. Since $f$ is multilinear, we have
\[
\sum_{j=1}^{t} f(\beta_1, \ldots, \phi(\beta_j), \ldots, \beta_t) = \frac{1}{2}[\phi(\alpha)\alpha^{-1} + \alpha\gamma\alpha^{-1}, f(\beta_1, \ldots, \beta_t)]
\]
for all $\beta_i \in D$. Applying Kharchenko's theorem [11] we have that
\[
\sum_{j=1}^{t} f(x_1, \ldots, y_j, \ldots, x_t) = \frac{1}{2}[\phi(\alpha)\alpha^{-1} + \alpha\gamma\alpha^{-1}, f(x_1, \ldots, x_t)]
\]
for all $x_i, y_i \in D$. In particular, taking $y_1 = \cdots = y_t = 0$ we get
\[
\frac{1}{2}[\phi(\alpha)\alpha^{-1} + \alpha\gamma\alpha^{-1}, f(x_1, \ldots, x_i)] = 0 \quad \text{for all } x_i \in D.
\]
Thus (4) is reduced to $\sum_{j=1}^{t} f(x_1, \ldots, y_j, \ldots, x_t) = 0$ for all $x_i, y_i \in D$. So in particular $f(x_1, \ldots, x_t) = 0$ for all $x_i \in D$, a contradiction. This completes the proof of the theorem.

With the Main Theorem in hand, the following question is naturally raised: Let $R$ be a semiprime $K$-algebra and $f(x_1, \ldots, x_t)$ a monic polynomial over $K$. Suppose that $d$ is a derivation of $R$ such that $d(f(x_1, \ldots, x_t)) = 0$ for all $x_i \in R$. Then what can we say about the structure of $R$?

We conclude this paper with a precise description for the above question.
Theorem 2. Let \( R \) be a semiprime \( K \)-algebra with center \( Z \), \( Q \) the Martindale two-sided quotient ring of \( R \), and \( f(x_1, \ldots, x_t) \) a monic polynomial over \( K \). Suppose that \( d(f(x_1, \ldots, x_t)) \in Z \) for all \( x_1, \ldots, x_t \in R \). Then there is a ring decomposition \( Q = Q_1 \oplus Q_2 \oplus Q_3 \) satisfying

\[
\begin{align*}
(I) & \quad d(Q_1) = 0, \\
(II) & \quad Q_2 \text{ satisfies } S_4, \text{ the standard polynomial of degree } 4, \text{ and} \\
(III) & \quad f(x_1, \ldots, x_t) \text{ is a central polynomial for } Q_3. 
\end{align*}
\]

**Proof.** Denote by \( C \) the extended centroid of \( R \); then \( Z(Q) = C \). It is well known that \( d \) can be uniquely extended to \( Q \). By [12, Theorem 3] \( Q \) and \( R \) satisfy the same differential identities. Thus we have \( d(f(x_1, \ldots, x_t)) \in C \) for all \( x_i \in Q \). Let \( \mathcal{M} \) be any maximal ideal of \( B \), the complete Boolean algebra of idempotents of \( C \) [2]. Then \( \mathcal{M}Q \) is a \( d \)-invariant prime ideal of \( Q \). Let \( \overline{d} \) denote the canonical derivation of \( \overline{Q} = Q/\mathcal{M}Q \) induced by \( d \). Note that \( Z(\overline{Q})/\mathcal{M}Q \cong C/\mathcal{M}C \). Thus \( \overline{d}(f(x_1, \ldots, x_t)) \in (C + \mathcal{M}Q)/\mathcal{M}Q \) for all \( x_i \in \overline{Q} \). It follows from [8, Theorem 3; Lemma 6; 10, Lemma 2] that either \( f(x_1, \ldots, x_t) \) is a central polynomial for \( \overline{Q} \), or \( \overline{Q} \) satisfies \( S_4 \), or \( \overline{d} = 0 \). Thus we have \( d(Q)QS_4(z_1, z_2, z_3, z_4)[f(x_1, \ldots, x_t), y] \subseteq \mathcal{M}Q \) for all \( x_i, z_i \in Q \). But since \( \{\mathcal{M}Q/\mathcal{M} \text{ is any maximal ideal of } B\} = 0 \), we obtain

\[
d(Q)QS_4(z_1, z_2, z_3, z_4)[f(x_1, \ldots, x_t), y] = 0 \quad \text{for all } x_i, y, z_i \in Q.\]

By [2, Point 2] there exists an idempotent \( h \in C \) such that \( \{\alpha \in C|ad(Q) = 0\} = hC \). Then \( d(hQ) = hd(Q) = 0 \) and

\[
S_4(z_1, z_2, z_3, z_4)(1 - h)Q[f(x_1, \ldots, x_t), y] = 0
\]

for all \( x_i, z_i, y \in (1 - h)Q \). But \( (1 - h)Q \) is still an orthogonally complete ring; there exists an idempotent \( g \in (1 - h)C \) such that

\[
\{\beta \in (1 - h)C|\beta S_4(z_1, z_2, z_3, z_4) = 0 \text{ for all } z_i \in (1 - h)Q\} = gC.
\]

So \( gQ \) satisfies \( S_4 \) and \( (1 - h)(1 - g)Q \) satisfies \( [f(x_1, \ldots, x_t), y] \). Now set \( Q_1 = hQ \), \( Q_2 = gQ \), and \( Q_3 = (1 - h)(1 - g)Q \). Then \( Q = Q_1 \oplus Q_2 \oplus Q_3 \) as desired. This completes the proof.

**References**


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