QUADRATIC AND QUASI-QUADRATIC FUNCTIONALS

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Abstract. In this note we show how Jordan \*-derivations arise as a "measure" of the representability of quasi-quadratic functionals by sesquilinear ones. Our main result can be considered as an extension of the Jordan-von Neumann characterization of pre-Hilbert space.

1. Introduction

Let $M$ be a module over a \*-ring $R$. A mapping $S: M \times M \to R$ is called a sesquilinear functional if it is linear in the first argument and antilinear in the second argument:

(1) $S(ax + by, z) = aS(x, z) + bS(y, z), \quad x, y, z \in M, \ a, b \in R,$
(2) $S(x, ay + bz) = S(x, y)a^* + S(x, z)b^*, \quad x, y, z \in M, \ a, b \in R.$

In the special case when $R$ is a commutative ring with the trivial involution $a^* = a$, the relation (2) can be rewritten as $S(x, ay + bz) = aS(x, y) + bS(x, z)$. In this case the mapping $S$ is called bilinear.

A quadratic functional $Q$ on $M$ is defined as the composition of some sesquilinear functional from $M \times M$ to $R$ with the diagonal injection of $M$ into $M \times M$; that is, $Q(x) = S(x, x)$, where $S$ is sesquilinear. There is something inappropriate about defining a quadratic functional which is a function of one variable in terms of a sesquilinear functional which involves two variables. This raises the question of what requirements can be imposed on a mapping from $M$ to $R$ to define the set of all quadratic functionals. The best-known identities satisfied by quadratic functionals are the parallelogram law

(3) $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \quad x, y \in M,$

and the homogeneity equation

(4) $Q(ax) = aQ(x)a^*, \quad x \in M, \ a \in R.$

A mapping $Q: M \to R$ satisfying these two identities is called a quasi-quadratic functional. In the special case that $R$ is a commutative ring with the trivial involution the relation (4) can be rewritten as $Q(ax) = a^2Q(x)$. 

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It seems natural to ask when quasi-quadratic functionals are in fact quadratic functionals. In other words, given a quasi-quadratic functional $Q$, does there exist a sesquilinear functional $S$ such that $Q(x) = S(x, x)$? In 1963, Halperin in his lectures on Hilbert spaces posed this problem for the special case that $M$ is a vector space over $F \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}$. Here, $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and the field of complex numbers respectively, while $\mathbb{H}$ denotes the skew-field of quaternions. In 1964, Kurepa [4] obtained the general form of quasi-quadratic functionals defined on a vector space over $\mathbb{R}$. In particular, he showed that there exist quasi-quadratic functionals which cannot be represented by bilinear functionals. In 1966, Gleason [2] generalized this result to vector spaces $V$, $\dim V \geq 2$, over an arbitrary field $F$, not of characteristic 2, and with the trivial involution. He proved that all quasi-quadratic functionals on $V$ are quadratic if and only if all additive derivations on $F$ are zero. The same result holds for quasi-quadratic functionals defined on a module over a commutative ring $R$ with the trivial involution in which 2 is a unit. This result follows from [1, Theorem 3]. It should be mentioned that in this commutative case with the trivial involution the result of Jordan and von Neumann [3] implies that for each quasi-quadratic functional $Q$ the mapping $S$ defined by

$$4S(x, y) = Q(x + y) - Q(x - y)$$

is symmetric and biadditive and $Q(x) = S(x, x)$ (see [2]). Thus, the above-mentioned results imply that $S$ is homogeneous in both variables if and only if all additive derivations on $R$ are zero.

In 1965, Kurepa [5] gave a positive answer to Halperin’s problem for quasi-quadratic functionals defined on a vector space $V$ over $F \in \{ \mathbb{C}, \mathbb{H} \}$. In 1984, Vukman [9] posed the problem of representability of quasi-quadratic functionals by sesquilinear ones on modules over complex $*$-algebras. This problem was treated in [6-11]. The complete solution was given in [7]. It was proved that if $Q$ is a quasi-quadratic functional on a module over a complex $*$-algebra with an identity element, then the mapping $S$ defined by

$$S(x, y) = \frac{1}{4}(Q(x + y) - Q(x - y)) + \frac{i}{4}(Q(x + iy) - Q(x - iy))$$

is the unique sesquilinear functional satisfying $Q(x) = S(x, x)$. This result is an extension of the Jordan-von Neumann theorem [3] which characterises pre-Hilbert space among all normed spaces.

A mapping $J$ defined on a $*$-ring $R$ is called a Jordan $*$-derivation if it is additive and satisfies

$$J(a^2) = aJ(a) + J(a)a^*.$$ 

We shall denote by $\mathcal{J}$ the set of all Jordan $*$-derivations on $R$. Over a commutative ring with the trivial involution in which 2 is not a zero divisor, the set of all Jordan $*$-derivations is equal to the set of all additive derivations [1]. A mapping $J_a: R \to R$, $a \in R$, defined by $J_a(b) = ba - ab^*$ will be called an inner Jordan $*$-derivation. In [8] it was proved that the representability of quasi-quadratic functionals by sesquilinear functionals on modules over a real Banach $*$-algebra $A$ with an identity element depends on the existence of Jordan $*$-derivations on $A$ which are not inner. The proof of this result given in [8] uses the fact that Banach algebras have enough invertible elements. It is the purpose of this note to extend this result to quasi-quadratic functionals.
defined on modules over arbitrary \(*\)-rings. In this general setting it is impossible to find a relation (similar to (5) in the commutative case) telling us how to recover from a quadratic functional \(Q\) a sesquilinear functional \(S\) satisfying \(Q(x) = S(x, x)\).

2. Statement of the results

**Main Theorem.** Let \(R\) be a \(*\)-ring with identity 1 such that \(2\) is a unit in \(R\). Assume that for every Jordan \(*\)-derivation \(J: R \to R\) there exists a unique \(a \in R\) such that \(J(b) = J_0(b) = ba - ab^*\), \(b \in R\). Then every quasi-quadratic functional \(Q\) defined on an arbitrary unitary \(R\)-module \(M\) is a quadratic functional.

Note that the uniqueness of \(a\) in the above theorem is equivalent to the statement that \(ba - ab^* = 0\) for all \(b \in R\) implies \(a = 0\). For the proof of the Main Theorem we shall need the following simple lemma.

**Lemma 1.** Let \(R\) be a \(*\)-ring with identity 1 such that \(ba - ab^* = 0\) for all \(b \in R\) implies \(a = 0\). If \(e_1, i = 1, 2, 3, 4\), are elements from \(R\) such that
\[
ae_1a^* + ae_2b^* + be_3a^* + be_4b^* = 0
\]
for all \(a, b \in R\) then \(e_i = 0\), \(i = 1, 2, 3, 4\).

The next theorem shows that the existence of noninner Jordan \(*\)-derivations yields the existence of quasi-quadratic functionals that cannot be represented by sesquilinear ones.

**Theorem 2.** Let \(R\) be a \(*\)-ring with identity 1 such that \(2\) is not a zero divisor. If \(J: R \to R\) is a Jordan \(*\)-derivation then the mapping \(Q: R \times R \to R\) given by \(Q((a, b)) = J(ba) - bJ(a) - J(a)b^*\) is a quasi-quadratic functional. If \(J\) is not inner then \(Q\) is not a quadratic functional.

A ring \(R\) is said to be a prime ring if \(aRb = \{0\}\) implies \(a = 0\) or \(b = 0\). We shall prove that the mapping \(F: R \to \mathcal{F}, F(a) = J_a\), is one-to-one if \(R\) is a noncommutative prime ring. Thus, we shall prove the following result.

**Corollary 3.** Let \(R\) be a noncommutative prime \(*\)-ring with identity 1 such that \(2\) is a unit in \(R\). Then all Jordan \(*\)-derivations on \(R\) are inner if and only if every quasi-quadratic functional \(Q\) defined on an arbitrary unitary \(R\)-module \(M\) is a quadratic functional.

Next, we shall show that all the assumptions of the Main Theorem are satisfied if \(R\) is a complex \(*\)-algebra with an identity element. This together with the Main Theorem implies the following extension of the Jordan-von Neumann characterization of pre-Hilbert spaces (see [7]).

**Corollary 4.** Let \(R\) be a complex \(*\)-algebra with identity 1 and let \(M\) be a unitary \(R\)-module. Assume that \(Q: M \to R\) is a quasi-quadratic functional. Under these conditions the mapping \(S: M \times M \to R\) defined by the relation (6) is the unique sesquilinear functional satisfying \(Q(x) = S(x, x)\).

We shall conclude by giving an example of a Jordan \(*\)-derivation which is not inner.

**Example 5.** There exists a Jordan \(*\)-derivation on a finite-dimensional noncommutative real \(*\)-algebra with an identity element which is not inner.
3. Proofs

Proof of Main Theorem. Let $Q$ be a quasi-quadratic functional defined on a unitary $R$-module $M$. We shall divide our proof into two steps. First, we shall prove that if the restriction of $Q$ to each submodule of $M$ generated by two elements is a quadratic functional, then $Q$ is a quadratic functional on $M$. Our second step will be to prove that under the assumptions of the Main Theorem every quasi-quadratic functional defined on an arbitrary unitary $R$-module $M$ generated by two elements is a quadratic functional.

Step 1. Assume that the restriction of $Q$ to each submodule of $M$ generated by two elements is a quadratic functional. Let us choose arbitrary elements $x, y \in M$. We denote by $M_{x,y} = \{ax + by : a, b \in R\}$ the submodule of $M$ generated by $x$ and $y$. According to our assumption there exists a sesquilinear functional $S_{x,y}: M_{x,y} \times M_{x,y} \to R$ such that

$$Q(ax + by) = S_{x,y}(ax + by, ax + by)$$

$$= aS_{x,y}(x, x)a^* + aS_{x,y}(x, y)b^* + bS_{x,y}(y, x)a^* + bS_{x,y}(y, y)b^*, \quad a, b \in R.$$ 

Let us define a functional $S: M \times M \to R$ by $S(x, y) = S_{x,y}(x, y)$ for all $x, y \in M$.

In order to see that the mapping $S$ is well defined we assume that there exists another sesquilinear functional $T_{x,y}: M_{x,y} \times M_{x,y} \to R$ satisfying

$$Q(ax + by) = T_{x,y}(ax + by, ax + by)$$

$$= aT_{x,y}(x, x)a^* + aT_{x,y}(x, y)b^* + bT_{x,y}(y, x)a^* + bT_{x,y}(y, y)b^*, \quad a, b \in R.$$ 

Comparing this with (8) and using Lemma 1 we get that $S_{x,y}(x, y) = T_{x,y}(x, y)$. Thus, $S$ is well defined. Moreover, we have proved that

$$S_{x, y}(x, y) = S_x(x, x)$$

holds for all $x, y \in M$. Let $x, y, z$ be elements from $M$. Then we have $S_{x, y}(x, x) = Q(1x + 0y) = Q(1x + 0z) = S_{x, z}(x, x)$. In particular, we obtain $S_{x, z}(x, x) = S_{y, z}(x, x)$. This last relation implies together with (9) that (8) can be rewritten as

$$Q(ax + by) = aS(x, x)a^* + aS(x, y)b^* + bS(y, x)a^* + bS(y, y)b^*, \quad a, b \in R,$$

where $x, y$ are arbitrary elements from $M$. It follows that $Q(x) = S(x, x)$ is valid for all $x \in M$. In order to complete the first step of our proof we must show that $S$ is a sesquilinear functional.

For arbitrary $x, y \in M$ and $a, b, c, d \in R$ we have

$$caS(x, x)a^*c^* + caS(x, y)b^*d^* + dbS(y, x)a^*c^* + dbS(y, y)b^*d^*$$

$$= Q(cax + dby) = cS(ax, ax)c^* + cS(ax, by)d^* + dS(by, ax)c^* + dS(by, by)d^*.$$ 

Applying Lemma 1 we get $S(ax, by) = aS(x, y)b^*$. It remains to prove that $S$ is biadditive. Define

$$b_1 = Q(a_1x_1 + a_2x_2 + a_3x_3), \quad b_2 = Q(a_1x_1 + a_2x_2 - a_3x_3),$$
and
\[ b_3 = Q(a_1 x_1 - a_2 x_2 - a_3 x_3). \]
The parallelogram law (3) gives us
\[
\begin{align*}
  b_1 + b_2 &= 2Q(a_1 x_1 + a_2 x_2) + 2Q(a_3 x_3), \\
  -b_2 - b_3 &= -2Q(a_1 x_1 - a_3 x_3) - 2Q(a_2 x_2), \\
  b_1 + b_3 &= 2Q(a_1 x_1) + 2Q(a_2 x_2 + a_3 x_3).
\end{align*}
\]
Solving this system of equations and using (10) we obtain
\[
b_1 = \sum_{i,j=1}^3 a_i S(x_i, x_j) a_j^*.
\]
In particular, for arbitrary \( x, y, z \in M \) and \( a, b \in \mathbb{R} \) we have the relation
\[
Q(ax + ay + bz) = a(S(x, x) + S(y, x) + S(x, y) + S(y, y)) a^* \\
+ a(S(x, z) + S(y, z)) b^* \\
+ b(S(z, x) + S(z, y)) a^* + bS(z, z) b^*.
\]
On the other hand, using (10) we get that
\[
Q(a(x + y) + bz) = aS(x + y, x + y) a^* + aS(x + y, z) b^* \\
+ bS(z, x + y) a^* + bS(z, z) b^*.
\]
Comparing the two expressions for \( Q(ax + ay + bz) \) we obtain, using Lemma 1, the biadditivity of \( S \). Thus, under the assumptions of the Main Theorem, a quasi-quadratic functional \( Q \) on \( M \) is a quadratic functional if and only if its restriction to each submodule generated by two elements is a quadratic functional.

**Step 2.** Let \( M = \{ax + by : a, b \in \mathbb{R} \} \) be a unitary \( \mathbb{R} \)-module generated by \( x \) and \( y \). We have to prove that for a given quasi-quadratic functional \( Q: M \to \mathbb{R} \) there exists a sesquilinear functional \( S \) from \( M \times M \) to \( \mathbb{R} \) such that \( Q(z) = S(z, z) \) for all \( z \in M \).

Let us define a functional \( D: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by
\[
D(a, b) = Q(ax + by) - aQ(x)a^* - bQ(y)b^* - 2^{-1}(afb^* + bfa^*),
\]
where \( f = Q(x + y) - Q(x) - Q(y) \). We shall first prove that \( D \) is biadditive. Clearly, it is enough to prove that the functional \( E \) given by \( E(a, b) = Q(ax + by) - aQ(x)a^* - bQ(y)b^* \) is biadditive. Applying the parallelogram law (3) we get
\[
2E(a, b) + 2E(c, b) = 2Q(ax + by) + 2Q(cx + by) - 2Q(ax) - 2Q(cx) - 4Q(by)
\]
\[
= Q((a + c)x + 2by) + Q((a - c)x) - 2Q(ax) - 2Q(cx) - Q(2by)
\]
\[
= Q((a + c)x + 2by) - Q((a + c)x) - Q(2by) = E(a + c, 2b).
\]
Substituting \( c = 0 \) and using the obvious relation \( E(0, b) = 0 \) we obtain
\[
2E(a, b) = E(a, 2b).
\]
It follows from (12) and (13) that the mapping \( E \) is additive in the first argument. The same must be true for the functional \( D \). In the same way we prove that \( D \) is additive in the second argument.
It is not difficult to verify that (4) and (11) imply
\[ D(a, a) = 0, \quad a \in R, \]
and
\[ D(ca, cb) = cD(a, b)c^*, \quad a, b, c \in R. \]
Using these two relations and biadditivity of \( D \) we shall prove that the mapping \( J: R \to R \) given by \( J(a) = D(a, 1) \) is a Jordan \(*\)-derivation satisfying
\[ (14) \quad D(a, b) = J(ab) - aJ(b) - J(b)a^*, \quad a, b \in R. \]
Clearly, \( J \) is additive. For arbitrary \( a, b, c, d \in R \) we have
\[
\begin{align*}
& aD(b, c)a^* + D(db, ac) + D(ab, dc) + dD(b, c)d^* \\
& = D((a + d)b, (a + d)c) = (a + d)D(b, c)(a + d)^* \\
& = aD(b, c)a^* + dD(b, c)a^* + aD(b, c)d^* + dD(b, c)d^*,
\end{align*}
\]
which yields
\[ D(db, ac) + D(ab, dc) = dD(b, c)a^* + aD(b, c)d^*. \]
Putting \( c = d = 1 \) we get \( D(b, a) + J(ab) = J(b)a^* + aJ(b) \). As \( D(a, a) = 0 \) implies \( D(a, b) = -D(b, a) \), we have proved that (14) is valid. Replacing \( a \) in this relation by \( ba \) we see that
\[ bJ(a)b^* = J(bab) - baJ(b) - J(b)a^*b^* \]
holds for all \( a, b \in R \). Putting \( a = 1 \) and using \( J(1) = 0 \) we finally get \( J(b^2) = bJ(b) + J(b)b^* \) for all \( b \in R \).

According to our assumptions, \( J \) is an inner Jordan \(*\)-derivation. Thus, we can find an element \( g \in R \) such that \( J(a) = ag - ga^* \) is valid for all \( a \in R \). It follows from (14) that
\[ D(a, b) = agb^* - bga^*, \quad a, b \in R. \]
Applying (11) one can easily see that
\[ Q(ax + by) = ae_{11}a^* + ae_{12}b^* + be_{21}a^* + be_{22}b^*, \quad a, b \in R, \]
where \( e_{11} = Q(x), e_{12} = g + 2^{-1}f, e_{21} = 2^{-1}f - g, \) and \( e_{22} = Q(y) \). We define \( S: M \times M \to R \) by
\[ S(ax + by, cx + dy) = ae_{11}c^* + ae_{12}d^* + be_{21}c^* + be_{22}d^*, \quad a, b, c, d \in R. \]
In order to see that \( S \) is well defined we choose \( a_1, a_2 \in R \) such that \( a_1x + a_2y = 0 \). For arbitrary elements \( b_1, b_2 \in R \) we have
\[
\begin{align*}
\sum_{i, j=1}^2 b_ie_{ij}b_j^* &= Q(b_1x + b_2y) = Q((a_1 + b_1)x + (a_2 + b_2)y) \\
&= \sum_{i, j=1}^2 (a_i + b_i)e_{ij}(a_j^* + b_j^*) \\
&= \sum_{i, j=1}^2 a_ie_{ij}a_j^* + \sum_{i, j=1}^2 a_ie_{ij}b_j^* + \sum_{i, j=1}^2 b_ie_{ij}a_j^* + \sum_{i, j=1}^2 b_ie_{ij}b_j^*.
\end{align*}
\]
It follows from \( 0 = Q(ax + ay) = \sum_{i,j=1}^{2} a_i e_{ij} a_j^* \) that
\[
\sum_{i,j=1}^{2} a_i e_{ij} b_j^* + \sum_{i,j=1}^{2} b_i e_{ij} a_j^* = 0.
\]

Putting \( b_1 = 1 \) and \( b_2 = 0 \) we get \( p + q = 0 \), where
\[
p = a_1 e_{11} + a_2 e_{21}, \quad q = e_{11} a_1^* + e_{12} a_2^*.
\]

On the other hand, if we set in (15) \( b_1 = c \) and \( b_2 = 0 \), we obtain \( pc^* + cq = 0 \). Together with \( cq + cp = 0 \) this implies \( cp - pc^* = 0 \) for all \( c \in A \). It follows that \( p = q = 0 \), or
\[
S(ax + ay, x) = 0 = S(x, ax + ay).
\]

In a similar way we get
\[
S(ax + ay, y) = 0 = S(y, ax + ay).
\]

Thus, \( S \) is well defined. Clearly, it is a sesquilinear functional satisfying \( Q(z) = S(z, z) \) for all \( z \in M \). This completes the proof.

Proof of Lemma 1. Putting \( a = 1 \) and \( b = 0 \) we get \( e_1 = 0 \). Similarly, we obtain \( e_4 = 0 \). Substituting \( a = b = 1 \) we see that \( e_2 = -e_3 \). Substituting once again \( b = 1 \) we get that \( ae_2 - e_2a^* = 0 \) is valid for all \( a \in R \). Thus, \( e_2 = e_3 = 0 \). This completes the proof.

Proof of Theorem 2. It is easy to verify that \( Q \) satisfies the parallelogram law (3). In order to see that also the homogeneity law (4) is fulfilled we must show that every Jordan *-derivation \( J : R \to R \) satisfies
\[
J(cbca) = cbJ(ca) + J(ca)b^*c^* + cJ(ba)c^* - cbJ(a)c^* - cJ(b)c^*.
\]

for all \( a, b, c \in R \). For this purpose first replace \( a \) by \( a + b \) in (7) to get
\[
J(ab) + J(ba) = bJ(a) + aJ(b) + J(a)b^* + J(b)a^*
\]

for all \( a, b \in R \). Consider now \( d = J(a(ab + ba) + (ab + ba)a) \). Using (17) we see that
\[
d = aJ(ab + ba) + (ab + ba)J(a) + J(ab + ba)a^* + J(a)(b^*a^* + a^*b^*)
\]
\[
= 2abJ(a) + a^2J(b) + aJ(a)b^* + 2aJ(b)a^* + baJ(a)
\]
\[
+ bJ(a)a^* + 2J(a)b^*a^* + J(b)a^2 + J(a)a^*b^*.
\]

On the other hand,
\[
d = 2J(aba) + J(a^2b) + J(ba^2)
\]
\[
= 2J(ab) + bJ(a^2) + a^2J(b) + J(a^2)b^* + J(b)a^2
\]
\[
= 2J(aba) + baJ(a) + bJ(a)a^* + a^2J(b) + aJ(a)b^* + J(a)a^*b^* + J(b)a^2.
\]

Comparing the two expressions for \( d \) we arrive at
\[
J(ab) = J(a)b^*a^* + aJ(b)a^* + abJ(a), \quad a, b \in R.
\]

Replacing \( a \) in (18) by \( a + c \) we obtain
\[
J(abc + cba) = J(a)b^*c^* + aJ(b)c^* + abJ(c) + J(c)b^*a^*
\]
\[
+ cJ(b)a^* + cbJ(a), \quad a, b, c \in R.
\]
Applying (18) and (19) we get
\[ J(cbca) = J(cb(ca) + (ca)bc) - J(ab)c \]
\[ = cbJ(ca) + J(ca)b^*c^* + c(J(b)a^* + aJ(b) - J(ab))c^*. \]

Applying (17) we get (16). Thus, we have proved that \( Q \) is a quasi-quadratic functional.

Assume now that \( J \) is not inner. If there is a sesquilinear functional \( S \) which generates \( Q \), then \( S \) is of the form \( S((a, b), (c, d)) = aed^* + bfc^* \) for some \( e, f \in R \). The relation \( Q((a, b)) = S((a, b), (a, b)) \) with \( b = 1 \) gives us \( J(a) = -ae - fa^* \). Since \( J(1) = 0 \), we have \( e = -f \), so that \( J \) is an inner Jordan *-derivation. This contradiction completes the proof.

**Proof of Corollary 3.** Let us first assume that all Jordan *-derivations on \( R \) are inner. We claim that \( J_a = 0 \), \( a \in R \), implies \( a = 0 \). Indeed, for such an \( a \) we have
\[ ba = ab^* \]
for all \( b \in R \). Replacing \( b \) by \( bc \) and applying (20) two times we get
\[ (bc - cb)a = 0. \]
Substituting \( c = dc \) in (21) we obtain \( (bd - dcb)a = 0 \), which can be rewritten as
\[ (bd - db)ca + d(bc - cb)a = 0 \]
where \( b, c, d \) are arbitrary elements from \( R \). The second term is zero by (21). As \( R \) is noncommutative and prime, we have necessarily \( a = 0 \). Using the Main Theorem one can complete the proof of the “if part”. Theorem 2 shows that the converse is also true.

**Proof of Corollary 4.** Substituting \( a = ia \) and \( b = i \) in (17) we prove that every Jordan *-derivation on \( R \) is inner. From \( J_a(i) = 2ia \) it follows that \( a \neq 0 \) implies that \( J_a \) is nonzero. Using the Main Theorem one can complete the proof.

**Verification of Example 5.** Let \( R \) be a real *-algebra consisting of elements \( \lambda + u\mu \), where \( \lambda \) and \( \mu \) are complex numbers. We define the operations by
\[ t(\lambda + u\mu) = t\lambda + u(t\mu) \] for real \( t \), \( (\lambda_1 + u\mu_1) + (\lambda_2 + u\mu_2) = (\lambda_1 + \lambda_2) + u(\mu_1 + \mu_2) \), \( (\lambda_1 + u\mu_1)(\lambda_2 + u\mu_2) = \lambda_1\lambda_2 + u(\mu_1\lambda_2 + \lambda_1\mu_2) \) and the involution by \( (\lambda + u\mu)^* = \overline{\lambda} - u\mu \).

There exists a nontrivial and therefore discontinuous additive derivation on \( R \), that is, an additive function \( f: \mathbb{R} \to \mathbb{R} \) satisfying \( f(ts) = tf(s) + sf(t) \) for all pairs \( t, s \in \mathbb{R} \) (see [12]). Putting \( D(s + it) = f(s) - if(t) \) we get a function \( D: \mathbb{C} \to \mathbb{C} \) which is additive and satisfies \( D(\lambda^2) = 2\lambda D(\lambda) \). It is not difficult to verify that the mapping \( J: R \to R \) given by \( J(\lambda + u\mu) = uD(\lambda) \) is a Jordan *-derivation. However, it is discontinuous and therefore noninner.

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References


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