SOME INEQUALITIES FOR SUBMARKOVIAN GENERATORS 
AND THEIR APPLICATIONS TO THE PERTURBATION THEORY 

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ABSTRACT. We characterize the domain in $L^p$-space of generators of submarkovian semigroups in terms of the form domain in $L^2$ and give the corresponding inequality. Using this inequality we obtain a criterion for the formal difference $A - B$ of such generators to be a generator of a contraction semigroup in $L^p$. The conditions on perturbation are expressed in terms of forms, i.e., in $L^2$-terms.

The following elementary inequality with application to the generators of submarkovian semigroups was found by Stroock [S, CKuS] (see also [V]).

For any $p > 1$ and $0 < s, t < \infty$

$$4 \frac{p-1}{p^2} (s^{p/2} - t^{p/2})^2 \leq (s - t)(s^{p-1} - t^{p-1}) \leq (s^{p/2} - t^{p/2})^2.$$ 

The proof is quite simple:

$$(s - t)(s^{p-1} - t^{p-1}) = s^p + t^p - st(s^{p-2} + t^{p-2})$$

$$\leq s^p + t^p - 2st\sqrt{s^{p-2}t^{p-2}} = (s^{p/2} - t^{p/2})^2.$$ 

On the other hand,

$$\frac{4}{p^2} (s^{p/2} - t^{p/2})^2 = \left(\int_t^s z^{p/2-1} dz\right)^2 \leq \left|\int_t^s dz\right| \cdot \left|\int_t^s z^{p-2} dz\right| \cdot (p-1)^{-1}.$$ 

In this paper we give two generalizations of this inequality and apply them to the generators of submarkovian semigroups. We get a characterization for the domain of the generator in the $L^p$-space in terms of the corresponding quadratic form in the $L^2$-space. We use the two-sided inequality obtained for the development of the perturbation theory for the generators in the $L^p$-space. The conditions on perturbation are expressed in terms of form-bounded perturbations, i.e., in $L^2$-terms. So in a sense, in our Theorem 2 below, we give an extension of the KLMN-theorem (see [RSi, K]) to the $L^p$-spaces.

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Lemma 1. For any $p \geq 1$ and $s, t \in \mathbb{R}$ the following inequality holds true:

$$4^{\frac{p-1}{p^2}}(s|s|^{p/2-1} - t|t|^{p/2-1})^2 \leq (s - t)(s|s|^{p-2} - t|t|^{p-2}) \leq a(p)(s|s|^{p/2-1} - t|t|^{p/2-1})^2$$

where

$$a(p) = \sup_{x \in [0, 1]} \frac{(x^{1/p} + 1)(x^{1/p'} + 1)}{(x^{1/2} + 1)^2}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

$a(1) = 2$, $a(2) = 1$, $1 \leq a(p) \leq 2$, $\forall p > 1$. (Moreover, if $s$ and $t$ are of the same sign, then $a(p)$ can be changed by 1.)

The proof of the left-hand side of the inequality is the same as that of Stroock's. The right-hand side is equivalent to

$$\frac{(s - t)(s|s|^{p-2} - t|t|^{p-2})}{(s|s|^{p/2-1} - t|t|^{p/2-1})^2} \leq a(p).$$

It is easy to see that

$$\sup_{s, t \in \mathbb{R}} \frac{(s - t)(s|s|^{p-2} - t|t|^{p-2})}{(s|s|^{p/2-1} - t|t|^{p/2-1})^2} = \sup_{x \in [0, 1]} \frac{(x^{1/p} + 1)(x^{1/p'} + 1)}{(x^{1/2} + 1)^2}.$$

Let us point out some properties of the function $a(p)$ which are needed further. Firstly, $a(p) = a(p')$; secondly, $a(p)$ decreases on $(1, 2)$ and increases on $(2, \infty)$.

Now we give some definitions and explanations.

Let $(M, \mu)$ be a measurable space with the $\sigma$-finite measure $\mu$. We use the following notation: $L^p = L^p(M, \mu)$, $\| \cdot \|_p$ is the norm in $L^p$, and $$(f, g) = \int_M f(x)g(x) d\mu(x).$$

We say that $A$ is a generator of a submarkovian semigroup (submarkovian generator) if the following conditions are satisfied:

(i) $A$ is a nonnegative selfadjoint operator in $L^2$.

(ii) $\| e^{-tA}f \|_\infty \leq \| f \|_\infty$, $\forall f \in L^1 \cap L^\infty$.

(iii) $0 \leq f \in L^2 \Rightarrow e^{-tA}f \geq 0$ almost everywhere.

We say that the operator $B$ is form-bounded relative to $A$ and write $B \in PK_{\beta}(A)$ if $B$ is a selfadjoint operator in $L^2$, $\mathcal{D}(\|B\|^{1/2}) \supset \mathcal{D}(A^{1/2})$, and

$$\| B \|^{1/2} \varphi \|_2 \leq \beta \| A^{1/2} \varphi \|_2 + C(\beta) \| \varphi \|_2 \quad \forall \varphi \in \mathcal{D}(A^{1/2})$$

for some $\beta \in (0, 1)$, $C(\beta) \geq 0$.

Now let $A$ be a submarkovian generator. We can define the operator $A_p$ as a generator of the contraction semigroup in $L^p$:

$$(e^{-tA} \uparrow [L^2 \cap L^p])_{t \rightarrow -\infty} = e^{-tA_p} \quad (\text{the closure in } L^p);$$

$$T'_\infty =: (e^{-tA_\lambda})^* \quad (\text{the sign of the adjoint operator}).$$

By representation of a resolvent in terms of a semigroup, we have

$$(\lambda + A_p)^{-1}[L^2 \cap L^p] = (\lambda + A)^{-1}[L^2 \cap L^p], \quad \forall \lambda > 0.$$
Theorem 1. Let $A$ be a submarkovian generator. If $f = \Re f \in \mathcal{D}(A_{p})$ for some $p \in (1, +\infty)$ then $f[f]^{p-2} = g_{p} \in \mathcal{D}(A^{1/2})$ and the following inequality holds true:

$$4\frac{p-1}{p^2} \|A^{1/2}g_{p}\|_{2}^{2} \leq \langle A_{p}f, f[f]^{p-2} \rangle \leq a(p)\|A^{1/2}g_{p}\|_{2}^{2}$$

where $a(p)$ is from (1) and if $f \geq 0$, then $a(p) = 1$.

Proof. Let $T' = T'_{\infty}$. Set $P(t, \cdot, G) = T'_{t}1_{G}$, $G \in \mathcal{B}$, where $\mathcal{B}$ is the $\sigma$-algebra on $M$ and $1_{G}$ is the characteristic function of the set $G$. $P(t, \cdot, G)$ is a finitely additive set function on $\mathcal{B}$ and $P(t, \cdot, M) \leq 1$.

For any simple function $f = \sum_{i=1}^{k} c_{i}1_{G_{i}}$, where $\{G_{i}\}$ are disjoint sets of finite measure, $c_{i} \in \mathbb{R}^{1}$, let us define

$$\int_{M} P(t, \cdot, dy)f(y) = T'f = \sum_{i=1}^{k} c_{i}T'_{t}1_{G_{i}},$$

$f \in \mathcal{N}$, $\mathcal{N}$ is the set of simple functions. Then

$$T'v_{p} = \sum_{i=1}^{k} c_{i}|c_{i}|^{p-2}T'_{t}1_{G_{i}}, \quad T'g_{p} = \sum_{i=1}^{k} c_{i}|c_{i}|^{p/2-1}T'_{t}1_{G_{i}},$$

where $v_{p} = |f|^{p-2}$, $g_{p} = |f|^{p/2-1}$.

Let $\varepsilon_{t}(u, v) = \frac{1}{t} \langle (1 - T'_{t})u, v \rangle$. Because of selfadjointness of $A$ and consequently of symmetricity on $(x, y)$ of the finite additive product-measure $d\mu_{t}(x, y) = P(t, x, dy)d\mu(x)$, the following equalities are valid:

$$\langle T'f, v_{p} \rangle = \langle f, T'v_{p} \rangle = \frac{1}{2} \int d\mu(x) \int P(t, x, dy)(f(x)v_{p}(y) + f(y)v_{p}(x)),$$

$$\langle (T'_{t}1_{E})f, v_{p} \rangle = \langle 1_{E}, T'_{t}|f|^{p} \rangle = \frac{1}{2} \int d\mu(x) \int P(t, x, dy)(|f(x)|^{p} + |f(y)|^{p})$$

where $E$ is the support of $f$. Hence we obtain

$$\varepsilon_{t}(f, v_{p}) = \frac{1}{2t} \int d\mu(x) \int P(t, x, dy)(f(x) - f(y))(v_{p}(x) - v_{p}(y))$$

$$+ \frac{1}{t} \langle (1 - T'_{t}1_{E}), |f|^{p} \rangle ,$$

$$\varepsilon_{t}(g_{p}, g_{p}) = \frac{1}{2t} \int d\mu(x) \int P(t, x, dy)(g_{p}(x) - g_{p}(y))^{2}$$

$$+ \frac{1}{t} \langle (1 - T'1_{E}), |f|^{p} \rangle .$$

By Lemma 1 we have

$$4\frac{p-1}{p^2} \varepsilon_{t}(g_{p}, g_{p}) \leq \varepsilon_{t}(f, v_{p}) + \frac{1}{t} \left(4\frac{p-1}{p^2} - 1 \right) \langle (1 - T'1_{E}), |f|^{p} \rangle$$

and

$$\varepsilon_{t}(f, v_{p}) \leq a(p)\varepsilon_{t}(g_{p}, g_{p}).$$

So we get

$$4\frac{p-1}{p^2} \varepsilon_{t}(g_{p}, g_{p}) \leq \varepsilon_{t}(f, v_{p}) \leq a(p)\varepsilon_{t}(g_{p}, g_{p}).$$
Since the set \( \mathcal{J} \) is dense in \( \text{Re} \, L^p \), \( 1 < p < \infty \), \( v_p \in \text{Re} \, L^p' \), (3) holds true for all \( f = \text{Re} \, f \in L^p \).

Now let \( f = \text{Re} \, f \in \mathcal{D}(A_p) \). Then (3) and the equality

\[
\mathcal{D}(A_{1/2}) = \left\{ \psi \in L^2 : \sup_{t>0} e_t(\psi, \psi) < \infty \right\},
\]

which follows from the spectral theorem, yield \( g_p \in \mathcal{D}(A_{1/2}) \) and the left-hand side of (2) if we set \( t \downarrow 0 \). The right-hand side of (2) now follows from (3) and the left-hand side. □

**Remark.** Theorem 1 is the generalization of the corresponding results of Stroock [S, CKuS] and Varopoulos [V] in the sense of our assumptions on the measurable space \( \mathcal{M} \). Besides, the main inequality has been proved on the natural domain.

Let \( a(p) \) be defined by (1). For a fixed \( \beta \in (0, 1) \) the equation \( \beta a(p) = 4(p - 1)/p^2 \) has exactly two solutions \( t_1 \in (1, 2) \) and \( t_2 = t_1' \in (2, \infty) \), where \( t_1' = t_1/((t_1 - 1) \cdot \). This is a direct consequence of the above-mentioned properties of the function \( a(p) \).

Theorem 1 and different consequences are discussed in more detail in [LPSe]. Here we give only the application to the perturbation theory.

**Theorem 2.** Let \( A \) and \( B \) be generators of submarkovian semigroups and \( B \in PK_\beta(A) \). Then the form-difference \( A - B = C \) is well defined and the following inequality is valid:

\[
\|e^{-tC} f\|_p \leq e^{a(p)C(\beta)t} \|f\|_p \quad \forall p \in [t(\beta), t'(\beta)]
\]

\( \forall f \in L^2 \cap L^p \), where \( C(\beta) \) is from the condition \( PK_\beta(A) \), and \( t(\beta) = t_1 \) and \( t'(\beta) = t_2 \) are the corresponding roots of the equation \( \beta a(q) = 4(q - 1)/q^2 \), \( 1 < q < \infty \).

**Proof.** Let

\[
B_n = nB(B + n)^{-1} = n - n^2(B + n)^{-1}
\]

(Yosida approximation), where \( B_n \) is a bounded selfadjoint operator in \( L^2 \). It is easy to check that \( B_n \) is a generator of a submarkovian semigroup. Besides, \( B_n \leq B \) and \( B_n \in PK_\beta(A) \) with the same \( \beta \) and \( C(\beta) \). So the operator \( C_{p,n} = A_p - B_{n,p} \) with \( \mathcal{D}(C_{p,n}) = \mathcal{D}(A_p) \) is the generator of the quasi-contractive semigroup \( T_n' \) in \( L^p \), \( 1 < p < \infty \), \( \forall n = 1, 2, \ldots \). Due to Stein [St, p. 67] these semigroups are holomorphic on \( L^p \), \( 1 < p < \infty \).

Let \( u_n(t) = e^{-tC_{p,n}}f, f \in L^2 \cap L^p \). Then \( u_n(t) \in \mathcal{D}(C_{p,n}) \) for any \( t > 0 \) and \( -du_n(t)/dt = C_{p,n}u_n(t) \). Note that \( e^{-tC_{p,n}}[\text{Re} \, L^2] \subset \text{Re} \, L^2 \) (it is a consequence of \( B_n[\text{Re} \, L^2] \subset \text{Re} \, L^2 \) and the Trotter product formula). Now without loss of generality, we can assume \( f = \text{Re} \, f \); then \( u_n = \text{Re} \, u_n \).

Multiplying both sides of the equation \( -\frac{d}{dt}u_n = (A_p - B_{n,p})u_n \) by \( u_n|u_n|^{p-2} \), integrating over \( \mathcal{M} \), and using Theorem 1 twice (for the operator \( A_p \) and for...
the operators $B_{n,p}$ and condition $B_{n,2} \leq B \in PK_\beta(A)$ we obtain

$$
-\frac{1}{p} \frac{d}{dt} \|u_n\|_p^p = \langle (A_p - B_{n,p})u_n, u_n|u_n|^{p-2} \rangle \\
= \langle A_p u_n, u_n|u_n|^{p-2} \rangle - \langle B_{n,p} u_n, u_n|u_n|^{p-2} \rangle \\
\geq 4 \frac{p-1}{p^2} \|A^{1/2}(u_n|u_n|^{p/2-1})\|_2^2 - a(p) \|B_{n,p}^{1/2}(u_n|u_n|^{p/2-1})\|_2^2 \\
\geq \left( \frac{4(p-1)}{p^2} - \beta a(p) \right) \|A^{1/2}(u_n|u_n|^{p/2-1})\|_2^2 - C(\beta)a(p) \|u_n\|_p^p.
$$

Consequently, for any $p \in [t(\beta), t'(\beta)]$

$$
\frac{d}{dt} \|u_n\|_p^p \leq pC(\beta)a(p) \|u_n\|_p^p.
$$

Thus $\|u_n(t)\|_p \leq e^{C(\beta)a(p)t} \|u_n(0)\|_p$ or

(5) $\|e^{-t(A-B)}f\|_p \leq e^{C(\beta)a(p)t} \|f\|_p \quad \forall f \in L^2 \cap L^p.$

Since $A-B \leq C_2, n \leq C_2, m$ provided $n \geq m$, then $e^{-tC_2, n} \to e^{-tC}$ strongly in $L^2$ [K, Chapter 6]. So that by (5) and by Fatou’s lemma we can pass to the limit in (5):

$$
\|e^{-t(A-B)}f\|_p \leq e^{C(\beta)a(p)t} \|f\|_p \quad \forall f \in L^2 \cap L^p. \quad \square
$$

Thus we have defined the operator $(A - B)_p$ in terms of form-boundedness and showed that this operator is a generator of quasi-contraction semigroups in $L^p$.

Remarks. 1. The sharp constant in the right-hand side of the inequality (2) can be less than $a(p)$ for certain submarkovian generators, for example, for potentials. Therefore the constant $a(p)$ in (4) could be replaced by $\lim_{n \to \infty} a_{B_n}(p)$.

2. The semigroup $e^{-t(A-B)}$ need not be positivity preserving. However, if $e^{-t(A-B)}$ are positivity preserving for sufficiently large $n$ then setting $u_n(t) = e^{-tC_{2,n}}f$, $f \in L^2 \cap L^p$, we can repeat the proof of Theorem 2 using inequality (2) with $\tilde{f} \geq 0$ and $a(p) = 1$. Instead of (5) we obtain

$$
\|e^{-t(A-B)}f\|_p \leq e^{C(\beta)a(p)t} \|f\|_p \quad \forall f \in L^2 \cap L^p.
$$

Thus taking into account the inequality $|e^{-tC_{2,n}}f| \leq e^{-tC_{2,n}}|f|$, we get

$$
\|e^{-t(A-B)}p\|_p \leq e^{C(\beta)t} \forall p \in [t_+(\beta), t'_+(\beta)],
$$

where $t_+(\beta) = 2/1 + \sqrt{1 - \beta}$, $t'_+(\beta) = 2/1 - \sqrt{1 - \beta}$.

3. Theorem 2 for the case of the Schrödinger operator $-\Delta - V$ and the sharpness of the dependence $t(\beta)$ as a function of $\beta$ was proved in [KoSe].

We now turn to the generalization of one of the inequalities proved in Theorem 1.

Lemma 2. Let $\varphi : \mathbb{R}_+^1 \to \mathbb{R}_+^1$ be a function such that

(i) $\varphi (z) = 0 \quad \forall z \in [0, b]$ for some $b \geq 0$;

(ii) $\varphi'(z) > 0 \quad \forall z > b$;
(iii) the function \( g_{\varphi}(z) \) is differentiable for \( z > b \) where

\[
g_{\varphi}(z) = \begin{cases} 
  z\varphi(z) - \kappa & \text{if } z \geq b, \\
  0 & \text{if } z < b,
\end{cases}
\]

\[
\phi(z) = \sqrt{\varphi'(z)}, \quad \kappa \equiv \varphi(z)|_{z=b};
\]

(iv) \( \sup_{z>b} (1 + z\varphi'(z)/\phi(z))^2 = c_{\varphi}^{-1} < \infty \).

Then for all \( t, s \in [b, +\infty) \)

\[
c_{\varphi}^{-1}(t-s)(\varphi(t) - \varphi(s)) \geq (g_{\varphi}(t) - g_{\varphi}(s))^2.
\]

Proof.

\[
(g_{\varphi}(t) - g_{\varphi}(s))^2 = \left( \int_s^t d g_{\varphi}(z) \right)^2 = \left( \int_s^t (z\varphi'(z) + \phi(z)) \, dz \right)^2
\]

\[
\leq \left( \int_s^t \left( 1 + \frac{z\varphi'(z)}{\phi(z)} \right)^2 \, dz \right) \left( \int_s^t \phi^2(z) \, dz \right)
\]

\[
\leq c_{\varphi}^{-1}(t-s)(\varphi(t) - \varphi(s)). \quad \square
\]

Theorem 3. Let \( \varphi \) and \( g_{\varphi} \) be the same functions as in Lemma 2. Let \( A \) be a generator of the submarkovian semigroup \( e^{-tA} \). If \( f = \text{Re} f \in \mathcal{D}(A_p) \) for some \( p \in [1, +\infty) \) and \( \varphi(|f|)A_p f \in L^1(M, \mu) \), then \( g_{\varphi}(|f|) \in \mathcal{D}(A^{1/2}) \) and

\[
c_{\varphi}\|A^{1/2}g_{\varphi}(|f|)\|_2^2 \leq \langle A_p f, (\text{sgn} f)\varphi(|f|) \rangle.
\]

The proof is almost the same as the proof of Theorem 1 when using the equality \( T_\infty^t f(x) = \int P(t, x, dy) f(y), \forall f \in L^\infty \), the evident inequality

\[
\langle A_p f, (\text{sgn} f)\varphi(|f|) \rangle \geq \lim_{t \to 0} \left\langle \frac{1 - t^t}{t} |f|, \varphi(|f|) \right\rangle,
\]

and Lemma 2. If we set \( \varphi(z) = z^{p-1} \), \( b = 0 \), then \( \varphi(z) = \sqrt{p - 1}z^{p/2-1} \), \( g_{\varphi}(z) = \sqrt{p - 1}z^{p/2} \), \( c_{\varphi}^{-1} = p^2/4 \). Hence we obtain the left-hand side of Stroock's inequality. If \( \varphi(z) = \ln z \), \( b = 1 \), then

\[
g_{\varphi}(z) = \begin{cases} 
  \sqrt{z} - 1 & \text{if } z \geq 1, \\
  0 & \text{if } z < 1
\end{cases}
\]

and \( c_{\varphi} = 4 \). Moreover, \( (s-t)(\ln s - \ln t) \geq 4(\sqrt{s} - \sqrt{t})^2 \forall s, t \geq 1 \). So from Theorem 3 we get that the conditions \( f = \text{Re} f \in \mathcal{D}(A_p) \) for some \( p \geq 1 \) and \( \ln_p |f| \cdot A_p f \in L^1 \) are sufficient to conclude \( 1_{|f|>1} \sqrt{|f|} \in \mathcal{D}(A^{1/2}) \) and the corresponding inequality holds true. This fact was a crucial tool in the investigation of the essential selfadjointness of the Schrödinger operator with negative form-bounded potential in the case of zero-bound [LSe]. It should be pointed out that for this case the analog of the right-hand side of the inequality of Theorem 1 cannot be obtained.

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