A NOTE ON A TRANSPLANTATION THEOREM OF KANJIN AND MULTIPLE LAGUERRE EXPANSIONS

S. THANGAVELU

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Abstract. By applying a transplantation theorem of Kanjin, a multiplier theorem and a Cesàro summability result are proved for multiple Laguerre expansions. In the one-dimensional case an improved version of the multiplier theorem is obtained.

Consider the normalised Laguerre functions \( \mathcal{L}_k^\alpha \), \( \alpha > -1 \), on \( \mathbb{R}_+ = (0, \infty) \) defined by

\[
\mathcal{L}_k^\alpha(t) = \left( \frac{\Gamma(k + 1)}{\Gamma(k + \alpha + 1)} \right)^{1/2} L_k^\alpha(t) e^{-t/2} t^{\alpha/2}
\]

where \( L_k^\alpha(t) \) are the Laguerre polynomials of type \( \alpha \). The functions \( \{\mathcal{L}_k^\alpha\} \) form a complete orthonormal system for \( L^2(\mathbb{R}_+) \). Recently, in [4] Kanjin studied the mapping properties of the operator \( T_\alpha^\beta \), which is defined as

\[
T_\alpha^\beta f = \sum_{k=0}^{\infty} (f, \mathcal{L}_k^\beta) \mathcal{L}_k^\alpha
\]

where \( (f, g) \) stands for the inner product in \( L^2(\mathbb{R}_+) \). For the operator \( T_\alpha^\beta \) he has proved the following result.

Theorem 1.1 (Kanjin). Let \( \alpha, \beta > -1 \) and \( \gamma = \min\{\alpha, \beta\} \). If \( \gamma \geq 0 \) then

\[
||T_\alpha^\beta f||_p \leq C||f||_p \quad \text{for} \ 1 < p < \infty.
\]

If \( -1 < \gamma < 0 \) then (1.3) is valid for \( p \) in the interval \((1 + \gamma/2)^{-1} < p < -2/\gamma\).

The above theorem is called a transplantation theorem for the following reason. Given a bounded sequence \( \lambda(k) \) we can define an operator \( M_\lambda^\alpha \) by setting

\[
M_\lambda^\alpha f = \sum_{k=0}^{\infty} \lambda(k) (f, \mathcal{L}_k^\alpha) \mathcal{L}_k^\alpha
\]
whenever \( f \) has the Laguerre expansion

\[
(1.5) \quad f = \sum_{k=0}^{\infty} (f, \mathcal{L}_k^\alpha) \mathcal{L}_k^\alpha.
\]

From the theorem, we can deduce the norm inequality

\[
(1.6) \quad \|M_\lambda^\alpha f\|_p \leq C\|f\|_p
\]

for any \( \alpha \) if we know (1.6) for a particular \( \alpha_0 \). This follows from the identity

\[(1.7) \quad T_\beta^\alpha M_\lambda^\alpha T_\alpha^\beta f = M_\lambda^\beta f.\]

As an application Kanjin proves the following result concerning \( M_\lambda^\alpha \).

**Theorem 1.2 (Kanjin).** Let \( \lambda(t) \) be a four times differentiable function on \((0, \infty)\) and satisfy

\[
(1.8) \quad \sup_{t>0} |t^k \lambda^{(k)}(t)| \leq c_k
\]

for \( k = 0, 1, 2, 3, 4 \). Then (1.6) is true for \( 1 < p < \infty \) if \( \alpha \geq 0 \) and for \( (1+\alpha/2)^{-1} < p < -2/\alpha \) if \(-1 < \alpha < 0\).

Theorem 1.2 is deduced by applying the transplantation theorem to the particular case \( \alpha = 0 \), which is proved by Dlugosz in [1]. Now the aim of this note is to prove an improved version of the above multiplier theorem and also to give applications to higher-dimensional Laguerre expansions.

Let \( \mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_j > 0 \text{ for all } j\} \), and consider for every \( \alpha \in \mathbb{R}_+^n \) and a multi-index \( m = (m_1, m_2, \ldots, m_n) \), the normalised Laguerre functions \( \mathcal{L}_m^\alpha \) on \( \mathbb{R}_+^n \) defined by

\[
(2.1) \quad \mathcal{L}_m^\alpha(x) = \prod_{j=1}^{n} \mathcal{L}_{m_j}^{\alpha_j}(x_j).
\]

They form a complete orthonormal system for \( L^2(\mathbb{R}_+^n) \), and the Laguerre expansion of a function \( f \) in \( L^p(\mathbb{R}_+^n) \) can be written as

\[
(2.2) \quad f = \sum_{m=0}^{\infty} (f, \mathcal{L}_m^\alpha) \mathcal{L}_m^\alpha
\]

where the sum is extended over all the multi-indices. Expansions of the above type have been studied by Dlugosz [1] when \( \alpha \) is a multi-index.

For the above series (2.2) we define the Cesàro means \( \sigma_N^\delta \) of order \( \delta \) by the equation

\[
(2.3) \quad \sigma_N^\delta f = \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_N^{\delta-k} \sum_{|m|=k} (f, \mathcal{L}_m^\alpha) \mathcal{L}_m^\alpha
\]

where \( A_N^\delta = \Gamma(k+\delta+1)/\Gamma(k+1) \) are the binomial coefficients. Given a function \( \lambda \) on \((0, \infty)\) we also define the multiplier operator \( M_\lambda^\alpha \) as

\[
(2.4) \quad M_\lambda^\alpha f = \sum_{k=0}^{\infty} \lambda(2k+n) \sum_{|m|=k} (f, \mathcal{L}_m^\alpha) \mathcal{L}_m^\alpha.
\]

For the operators (2.3) and (2.4) we prove the following two theorems.
Theorem 2.1. Let $\delta > \frac{1}{2}$. Then the uniform estimates
\begin{equation}
\|\sigma_N^\delta f\|_p \leq C\|f\|_p
\end{equation}
are valid iff $4n/(2n + 1 + 2\delta) < p < 4n/(2n - 1 - 2\delta)$.

Theorem 2.2. Assume that the function $\lambda$ satisfies the conditions
\begin{equation}
\sup_{t > 0} |t^k \lambda^{(k)}(t)| \leq c_k
\end{equation}
for $k = 0, 1, 2, \ldots, \nu$ where $\nu = n + 1$ if $n$ is odd and $\nu = n + 2$ if $n$ is even. Then for $1 < p < \infty$ we have
\begin{equation}
\|M_\alpha^\nu f\|_p \leq C\|f\|_p.
\end{equation}
In the case $n = 1$ we can take $\nu = 1$ in the hypothesis and (2.7) is valid for $\frac{4}{3} < p < 4$.

A slightly weaker form of Theorem 2.2 is proved in [1] when $\alpha$ is a multi-index. In that version one has $\nu = n + 3$ for all $n$. Theorem 2.1 is known when $n = 1$ and is due to Gorlich and Markett [3, 5].

For the Laguerre series (2.2) we also define the Riesz transforms $R_j$, $j = 1, 2, \ldots, n$, by the formula
\begin{equation}
R_j f = \sum_{m=0}^{\infty} (2m_j + 1)(2|m| + n)^{-1} (f, \mathcal{L}_m^\alpha) \mathcal{L}_m^\alpha.
\end{equation}
Riesz transforms for the Hermite and special Hermite expansions have been studied by the author in [9, 12]. For the above Riesz transforms (2.8) we prove

Theorem 2.3. For $1 < p < \infty$ all the Riesz transforms $R_j$ are bounded on $L^p(\mathbb{R}_+^n)$.

All three theorems will be proved by appealing to the $n$-dimensional version of Kanjin’s transplantation Theorem 1.1. For $\alpha, \beta$ in $\mathbb{R}_+^n$ we define $T_\alpha^\beta$ by
\begin{equation}
T_\alpha^\beta f = \sum_{m=0}^{\infty} (f, \mathcal{L}_m^{\beta}) \mathcal{L}_m^{\alpha}.
\end{equation}
Then, for $f$ in $C^\infty_0(\mathbb{R}_+^n)$ and $1 < p < \infty$,
\begin{equation}
\|T_\alpha^\beta f\|_p \leq C\|f\|_p.
\end{equation}
This follows from Theorem 1.1 by iteration.

In view of (2.10) Theorems 2.1, 2.2, and 2.3 will follow once we show that they are true in the particular case $\alpha = 0$. It will be shown in the next section that the case $\alpha = 0$ follows from known results on special Hermite expansions as a special case. The one-dimensional case of Theorem 2.2 when $\alpha = \frac{1}{2}$ will be deduced from the corresponding result on the Hermite expansions. This will be done in the last section.

Consider the functions $\psi_m(z)$ on $\mathbb{C}^n$ defined by
\begin{equation}
\psi_m(z) = \prod_{j=1}^{n} L_{m_j}(\frac{1}{2}|z_j|^2)e^{-|z_j|^2/4}
\end{equation}
where $L_k(t)$ are the Laguerre polynomials of type 0. The functions $\psi_m(z)$ are called special Hermite functions since they are related to the Hermite function $\Phi_m(x)$ on $\mathbb{R}^n$. This terminology is due to Strichartz [6]. In fact, one has

$$
\psi_m(z) = \int_{\mathbb{R}^n} e^{i\xi \cdot \zeta} \Phi_m \left( \xi + \frac{y}{2} \right) \Phi_m \left( \xi - \frac{y}{2} \right) \, d\xi
$$

where $z = x + iy$, $x, y \in \mathbb{R}^n$ (see [2]). Given $f$ on $\mathbb{C}^n$ we have the special Hermite expansion

$$
f(z) = (2\pi)^{-n} \sum_{m=0}^{\infty} f \times \psi_m(z)
$$

where the twisted convolution $f \times g$ of two functions is defined by

$$
f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{(i/2)\text{Im} z \cdot \overline{w}} \, dw.
$$

We can also write (3.3) in the form

$$
f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k^{n-1}(z)
$$

where $\varphi_k^{n-1}(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right)e^{-|z|^2/4}$. For all these facts we refer to [11].

For the special Hermite expansion let $C_N^\delta$ be the Cesàro means defined by

$$
C_N^\delta f = \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta \sum_{|m|=k} (f \times \psi_m).
$$

Given a function $\lambda$ on $(0, \infty)$ we also define a multiplier transform $T_\lambda$ by

$$
T_\lambda f = \sum_{k=0}^{\infty} \lambda(2k+n) \sum_{|m|=k} (f \times \psi_m).
$$

In [11] we proved

**Theorem 3.1.** Let $\delta > \frac{1}{2}$. Then for $f$ in $L^p(\mathbb{C}^n)$

$$
\|C_N^\delta f\|_p \leq C \|f\|_p
$$

holds if and only if $4n/(2n + 1 + 2\delta) < p < 4n/(2n - 1 - 2\delta)$.

Regarding $T_\lambda$ we have proved the following multiplier theorem in [10].

**Theorem 3.2.** Let $\lambda$ satisfy the hypothesis of Theorem 2.2. Then for $1 < p < \infty$ one has $\|T_\lambda f\|_p \leq C \|f\|_p$.

The case $\alpha = 0$ of Theorems 2.1 and 2.2 will be deduced from the above theorems in the following way. When $f$ is a radial function the twisted convolution $f \times \varphi_k^{n-1}$ becomes

$$
f \times \varphi_k^{n-1}(z) = \frac{k!(n-1)!}{(k+n-1)!} \left( \int_0^\infty f(r) \varphi_k^{n-1}(r) r^{2n-1} \, dr \right) \varphi_k^{n-1}(z)
$$
where \( \varphi_k^{n-1}(r) = \varphi_k^{n-1}(z) \) with \(|z| = r\). If \( f \) is a polyradial function, i.e., \( f(z_1, \ldots, z_n) = f(r_1, \ldots, r_n), r_j = |z_j| \), then in view of (3.8) and (3.1) one has

\[
(3.9) \quad f \times \psi_m = \left\{ \int_{\mathbb{R}^n_+} f(r_1, \ldots, r_n) \left( \prod_{j=1}^n \mathcal{L}_{m_j}(\frac{1}{2} r_j^2) \right) r_1 \cdots r_n \, dr_1 \cdots dr_n \right\} \psi_m.
\]

Therefore, one sees that

\[
(3.10) \quad f \times \psi_m(\sqrt{2} z) = (g, \mathcal{L}_m) \mathcal{L}_m(r)
\]

where \( g(r_1, \ldots, r_n) = f(\sqrt{2r_1}, \ldots, \sqrt{2r_n}) \). Therefore, \( C_N^\delta f \) becomes \( \sigma_N^\delta g \) and \( T_\alpha f \) becomes \( M_\alpha^0 g \); hence, the case \( \alpha = 0 \) of Theorems 2.1 and 2.2 follow.

The case \( \alpha = 0 \) of Theorem 3.3 follows from the fact (see [12]) that the Riesz transforms

\[
(3.11) \quad Sjf = \sum_{m=0}^{\infty} (2m_j + 1)(2|m| + n)^{-1} f \times \psi_m
\]

for the special Hermite expansions are bounded on \( L^p(\mathbb{C}^n), 1 < p < \infty \).

4

Consider the normalised Hermite functions \( h_k(x) \) on \( \mathbb{R} \). We also consider the Laguerre function \( \varphi_k^\omega \) of another type defined by, for \( \alpha \) real,

\[
(4.1) \quad \varphi_k^\omega(x) = \mathcal{L}_k^\omega(x^2)(2x)^{1/2}, \quad x \in \mathbb{R}_+.
\]

Then the Hermite functions \( h_k \) and \( \varphi_k^\omega \) are related by (see [7])

\[
(4.2) \quad h_{2k}(x) = (-1)^k \frac{1}{\sqrt{2}} \varphi_k^{-1/2}(x), \quad h_{2k+1}(x) = (-1)^k \frac{1}{\sqrt{2}} \varphi_k^{1/2}(x).
\]

Consider a multiplier transform \( M \) for the Hermite series defined by

\[
(4.3) \quad Mf(x) = \sum_{k=0}^{\infty} \lambda(k)(f, h_k)h_k(x).
\]

In [8] we proved

**Theorem 4.1.** Assume that \( \lambda \) is bounded and satisfies \( |\lambda'(t)| \leq C \) for all \( t > 0 \). Then \( M \) is bounded on \( L^p(\mathbb{R}), 1 < p < \infty \).

Since \( h_{2k} \) is even and \( h_{2k+1} \) is odd, by considering \( f \) to be odd we see that

\[
(4.4) \quad Mf(x) = \sum_{k=0}^{\infty} \lambda(2k + 1)(f, \varphi_k^{1/2})\varphi_k^{1/2}(x),
\]

and this is related to \( M_k^{1/2} \) in the following way. An easy calculation shows that

\[
(4.5) \quad (f, \varphi_k^{1/2}) = \frac{1}{\sqrt{2}} (g, \mathcal{L}_k^{1/2})
\]
where \( f(\sqrt{x})x^{-1/4} = g(x) \). Therefore,

\[
Mf(\sqrt{x})x^{-1/4} = 2 \sum_{k=0}^{\infty} \lambda(2k + 1)(g, L^1_k) L^1_k(x).
\]

If we know that for \( \frac{4}{3} < p < 4 \)

\[
\int_0^\infty |Mf(x)|^p x^{-p/2+1} \, dx \leq C \int_0^\infty |f(x)|^p x^{-p/2+1} \, dx
\]

then it follows that

\[
\int_0^\infty |M^{1/2} g(x)|^p \, dx \leq C \int_0^\infty |g(x)|^p \, dx;
\]

hence, the case \( n = 1, \alpha = \frac{1}{2} \) of Theorem 2.2 follows. We claim that (4.7) is true.

To prove the claim we recall the proof of Theorem 4.1. Let \( T' \) be the semigroup on \( L^p(\mathbb{R}) \) defined by

\[
T'f = \sum_{k=0}^{\infty} e^{-\alpha/(2k+1)} (f, h_k) h_k.
\]

For this semigroup we defined the \( g \) and \( g^* \) functions in the following way:

\[
(g(f, x))^2 = \int_0^\infty t \partial_t T' f(x) \, dt,
\]

\[
(g^*(f, x))^2 = \int_{-\infty}^\infty \int_0^\infty t^{1/2} (1 + t^{-1/2} |x-y|)^{-2} \partial_y T' f(y) \, dy \, dt.
\]

For the \( g \) and \( g^* \) functions we proved that

\[
C_1 \|f\|_p \leq \|g(f)\|_p \leq C_2 \|f\|_p, \quad 1 < p < \infty,
\]

\[
\|g^*(f)\|_p \leq C \|f\|_p, \quad p > 2.
\]

Under the assumption that \( |t\lambda'(t)| \) is bounded we verified that

\[
g(Mf, x) \leq C g^*(f, x),
\]

and in view of (4.12) and (4.13) this proved Theorem 4.1.

Therefore, in order to prove the weighted version we need to check that

\[
C_1 \|f\|_{p, w} \leq \|g(f)\|_{p, w} \leq C_2 \|f\|_{p, w}, \quad 1 < p < 4,
\]

\[
\|g^*(f)\|_{p, w} \leq C \|f\|_{p, w}, \quad 2 < p < 4,
\]

where \( \|f\|_{p, w} \) stands for the norm

\[
\|f\|_{p, w} = \left( \int_{-\infty}^{\infty} |f(x)|^p |x|^{-p/2+1} \, dx \right)^{1/p}.
\]

Thus we need weighted norm inequalities for the \( g \) and \( g^* \) functions.

In [8] we proved the \( L^p \) boundedness of \( g \) by applying singular integral theory. We identified \( g \) with a singular integral operator whose kernel takes
values in the Hilbert space $L^2(\mathbb{R}_+, t \, dt)$. When the weight function $w$ is in the Muckenhoupt class $A_p$ (see [13]) then we also have

$$
\int_{-\infty}^{\infty} |g(f)|^p w(x) \, dx \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x) \, dx.
$$

When $\frac{4}{3} < p < 4$, $w(x) = |x|^{-p/2+1}$ is in $A_p$; hence, the right-hand side inequality of (4.12)' is valid. We will now show that the reverse inequality is also valid.

From [8] we recall that we have the partial isometry

$$
\|g(f)\|_2 = \frac{1}{2} \|f\|_2;
$$

from this, by polarisation, we obtain

$$
\left| \int_{-\infty}^{\infty} f_1(x) \overline{f_2(x)} \, dx \right| = 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t \partial_t T_t f_1(x) \overline{T_t f_2(x)} \, dt \, dx.
$$

This gives the inequality

$$
\left| \int_{-\infty}^{\infty} f_1(x) \overline{f_2(x)} \, dx \right| \leq 4 \int_{-\infty}^{\infty} g(f_1, x) g(f_2, x) \, dx.
$$

Let us now take $h(x) = f_2(x)|x|^{-1/2+1/p}$ so that

$$
\left| \int_{-\infty}^{\infty} f_1(x)|x|^{-1/2+1/p} \overline{f_2(x)} \, dx \right|
\leq 4 \int_{-\infty}^{\infty} g(f_1, x)|x|^{-1/2+1/p} g(h, x)|x|^{-1/2+1/q} \, dx
$$

where $q$ is the index conjugate to $p$. An application of Holder's inequality gives

$$
\left( \int_{-\infty}^{\infty} |g(f_1, x)|^p |x|^{-p/2+1} \, dx \right)^{1/p} \left( \int_{-\infty}^{\infty} |g(h, x)|^q |x|^{-q/2+1} \, dx \right)^{1/q}.
$$

Applying the direct inequality (4.12)' to the second factor we get

$$
\int_{-\infty}^{\infty} |g(h, x)|^q |x|^{-q/2+1} \, dx \leq C \int_{-\infty}^{\infty} |f_2(x)|^q |x|^{-q/2+q/p+1-1/2} \, dx
\leq C \int_{-\infty}^{\infty} |f_2(x)|^q \, dx.
$$

In view of (4.20) and (4.21) the inequality (4.19) becomes

$$
\int_{-\infty}^{\infty} f_1(x)|x|^{-1/2+1/p} \overline{f_2(x)} \, dx \leq C \|g(f_1)\|_{p,w} \|f_2\|_q.
$$

Taking the supremum over all $f$ with $\|f_2\|_q \leq 1$ we obtain

$$
\int_{-\infty}^{\infty} |f_1(x)|^p |x|^{-p/2+1} \, dx \leq C \|g(f_1)\|_{p,w}.
$$

This completes the proof of (4.12)'.

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To establish the inequality (4.13)' we observe that

\[ \int_{-\infty}^{\infty} (g^*(f, x))^2 h(x) \, dx \leq \int_{-\infty}^{\infty} (g(f, x))^2 \Lambda h(x) \, dx \]

for every nonnegative function \( h \) where \( \Lambda h \) is the Hardy-Littlewood maximal function. If \( 2 < p < 4 \), let \( r = p/2 \) and \( s \) be the conjugate index of \( r \). Setting \( h_1(x) = h(x)|x|^{-1+1/r} \) we have

\[ \int_{-\infty}^{\infty} (g^*(f, x))^2 |x|^{-1+1/r} h(x) \, dx \]

(4.25) \[
\leq C \int_{-\infty}^{\infty} (g(f, x))^2 |x|^{-1+1/r} |x|^{1/2} \Lambda h_1(x) \, dx 
\]

\[ \leq C \left( \int_{-\infty}^{\infty} (g(f, x))^p |x|^{-p/2+1} \, dx \right)^{2/p} \left( \int_{-\infty}^{\infty} |x| (\Lambda h_1(x))^s \, ds \right)^{1/s} \]

by an application of Holder's inequality. Since \( s > 2 \), \( |x| \in A_s \); hence,

\[ \int_{-\infty}^{\infty} |x| (\Lambda h_1(x))^s \, ds \leq C \int_{-\infty}^{\infty} |h(x)|^s |x|^{-s+s/r+1} \, dx \]

(4.26)

\[ \leq C \int_{-\infty}^{\infty} |h(x)|^s \, dx. \]

Thus we have the inequality

\[ \int_{-\infty}^{\infty} (g^*(f, x))^2 |x|^{-1+1/r} h(x) \, dx \]

(4.27) \[
\leq C \left( \int_{-\infty}^{\infty} |f(x)|^p |x|^{-p/2+1} \, dx \right)^{2/p} \| h \|_s. \]

Taking the supremum over all \( h \) with \( \| h \|_s \leq 1 \) we obtain

\[ \int_{-\infty}^{\infty} (g^*(f, x))^p |x|^{-p/2+1} \, dx \leq C \int_{-\infty}^{\infty} |f(x)|^p |x|^{-p/2+1} \, dx. \]

(4.28)

This proves the inequality (4.13)'.

Therefore, in view of (4.12)', (4.13)', and (4.14) we obtain the weighted inequality

\[ \int_{-\infty}^{\infty} |Mf(x)|^p |x|^{-p/2+1} \, dx \leq C \int_{-\infty}^{\infty} |f(x)|^p |x|^{-p/2+1} \, dx \]

(4.29)

for \( \frac{4}{3} < p < 4 \), and this proves the multiplier theorem for \( \alpha = \frac{1}{2} \). By applying the transplantation theorem we complete the proof of Theorem 2.2 when \( n = 1 \).

References


10. ____, *Littlewood-Paley-Stein theory on $\mathbf{C}^n$ and Weyl multipliers*, Rev. Mat. Ibero Americana 6 (1990), 75–90.


Tata Institute for Fundamental Research Centre, Post Box No. 1234, Indian Institute of Science Campus, Bangalore, India

E-mail address: veluma@tifrbng.ernet.in