ON WEAK CONVERGENCE IN $H^1(\mathbb{R}^d)$

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Abstract. We prove that the a.e. convergence of a sequence of functions bounded in $H^1(\mathbb{R}^d)$ to a function in $L^1(\mathbb{R}^d)$ implies weak convergence.

Let $H^1(\mathbb{R}^d)$ denote the Hardy space of functions on $\mathbb{R}^d$ whose Poisson maximal function lies in $L^1$, and let $\text{BMO}(\mathbb{R}^d)$ denote the dual space [2] of functions of bounded mean oscillation. The space $\text{VMO}(\mathbb{R}^d)$, the closure of the Schwartz space $S'$ in $\text{BMO}(\mathbb{R}^d)$, is the predual of $H^1(\mathbb{R}^d)$. We answer a question of Lions and Meyer by proving the following result. The theorem is also valid for Martingale $H^1$, and the proof we give carries over directly to that setting.

Theorem. Suppose $\{f_n\}$ is a sequence of $H^1(\mathbb{R}^d)$ functions such that $\|f_n\|_{H^1} \leq 1$ for all $n$ and such that $f_n(x) \to f(x)$ for almost every $x \in \mathbb{R}^d$. Then $f \in H^1(\mathbb{R}^d)$, $\|f\|_{H^1} \leq 1$, and

$$\int_{\mathbb{R}^d} f_n \varphi \, dx \to \int_{\mathbb{R}^d} f \varphi \, dx$$

for all $\varphi \in \text{VMO}(\mathbb{R}^d)$.

Proof. We may suppose that $\|\varphi\|_{L^1}$, $\|\varphi\|_{L^\infty}$, $\|\partial \varphi / \partial x_j\|_{L^\infty} \leq 1$ and that support $\varphi$ is compact. Fix $\delta > 0$ and pick $\eta > 0$ so that $\eta \exp\{\delta^{-1}\} \leq \delta^{1+d}$ and, whenever $|E| \leq C \eta \exp\{\delta^{-1}\}$, $\int_E |f| \, dx \leq \delta$. This can be done because by Fatou's lemma $\|f\|_{L^1} \leq 1$. Now pick $n$ large enough so that

$$|E_n| = |\{x \in \text{support } \varphi : |f_n(x) - f(x)| > \eta\}| \leq \eta.$$

Define

$$\tau(x) = \max\{0, 1 + \delta \log M_{\chi_{E_n}}(x)\},$$

where $M(\cdot)$ denotes the Hardy-Littlewood maximal function. Then $0 \leq \tau \leq 1$, $\tau = 1$ a.e. $dx$ on $E_n$, and by a result due to Coifman and Rochberg [1], $\|\tau\|_{\text{BMO}} \leq C \delta$. By the weak $(1, 1)$ estimate for the maximal function [3],

$$|\text{support } \tau| \leq C |E_n| \exp\{\delta^{-1}\} \leq C \eta \exp\{\delta^{-1}\}.$$

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Consequently,
\[ \int_{\text{support } \tau} |f| \, dx \leq \delta. \]

We now write
\[ \left| \int_{\mathbb{R}^d} (f - f_n) \varphi \, dx \right| \leq \left| \int_{\mathbb{R}^d} (f - f_n) \varphi(1 - \tau) \, dx \right| + \left| \int_{\mathbb{R}^d} (f - f_n) \varphi \tau \, dx \right| \]
\[ \leq \eta \|\varphi\|_{L^1} + \int_{\text{support } \tau} |f\varphi| \, dx + \int_{\mathbb{R}^d} f_n \varphi \tau \, dx \]
\[ \leq \delta + \delta + C \|f_n\|_{H^1} \|\varphi\|_{\text{BMO}}. \]

The proof of the theorem will therefore be established as soon as we verify
\[ \|\varphi \tau\|_{\text{BMO}} \leq C \delta. \]

Since \( \|\varphi \tau\|_{L^1} \leq C \eta \exp\{\delta^{-1}\} \leq \delta^{1+d} \), it is sufficient to test the BMO norm over cubes of sidelength \( \leq \delta \). Let \( \tau_Q \) and \( \varphi_Q \) denote the mean values of \( \tau, \varphi \) over such a cube \( Q \), and note that \( |\varphi - \varphi_Q| \leq \delta \) on \( Q \). Therefore,
\[ \frac{1}{|Q|} \int_Q |\varphi - \varphi_Q \tau_Q| \, dx \leq \frac{1}{|Q|} \int_Q |\varphi - \varphi_Q \tau| \, dx + \frac{|\varphi_Q|}{|Q|} \int_Q |\tau - \tau_Q| \, dx \]
\[ \leq \delta + \|\tau\|_{\text{BMO}} \leq C \delta. \]

Standard limiting arguments show \( \|f\|_{H^1} \leq 1. \)

We remark that if \( f_n \) lie in the analytic Hardy space \( H^1_\alpha(\mathbb{R}^d) \), a proof can be given by using only \( H^\infty \) theory. By taking subsequences and looking at \( \log|f_n| \), one can find \( \varphi_m \in H^\infty \) such that \( \|\varphi_m\|_{H^\infty} \leq 1 \), \( \|f_n \varphi_m (1 + |x|^2)\|_{\infty} \leq m \), and \( \varphi_m(x) \to 1 \) a.e. \( dx \). Taking limits in \( n \), \( \|f \varphi_m\|_{H^1} \leq 1 \), and then taking limits in \( m \), \( \|f\|_{H^1} \leq 1 \). Returning to the original sequence, one applies Jensen's inequality to \( f_n - f \) to establish that \( f_n(z) - f(z) \to 0 \). This suffices to prove \( f_n \to f \) weakly.

Remark. The theorem is not abstract. In other words, if one attempts to use only the facts \( \|f_n\|_{L^1} \leq 1 \), \( f_n(x) \to f(x) \) a.e. \( dx \), and \( \mathcal{P} \) is dense in VMO, one will not arrive at a correct proof.

References

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