POLYNOMIAL APPROXIMATION
FOR A CLASS OF PHYSICAL RANDOM VARIABLES

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ABSTRACT. In white noise theory on Hilbert spaces, it is known that maps which are uniformly continuous around the origin in the S-topology constitute an important class of "physical" random variables. We prove that random variables having such a continuity property can be approximated in the gaussian measure by polynomial random variables. The proof relies on representing functions which are uniformly S-continuous around the origin as the composition of a continuous map with a Hilbert-Schmidt operator.

1. INTRODUCTION

Weak distribution theory and its associate white noise theory have provided, in recent years, an alternative approach to the usual Itô calculus for modeling physical systems in a noisy environment [1, 2]. The main concepts of this approach, which from the point of view of engineering application seems to be more useful than the conventional one, can briefly be outlined as follows.

Any physical signal taking values in a separable Hilbert space H and observed in a finite time interval $[0, T]$ has bounded variation and finite energy. Thus a natural space to model it is $\mathcal{H} = L_2([0, T]; H)$. Gaussian noise has a bandwidth much larger than the signal one. It follows that the white noise can be well modelled as the identity map on $\mathcal{H}$ endowed with the gaussian measure μ_G with identity covariance operator. The measure μ_G has the following characteristic functional:

$$C_{\mu_G}(h) = \int_{\mathcal{H}} e^{i[h, \omega]} d\mu_G(\omega) = e^{\|h\|^2/2}.$$

Then μ_G is defined only on the algebra $\mathcal{C}$ of cylinder sets of $\mathcal{H}$ and cannot be extended to the whole Borel σ-algebra $\mathcal{B}_{\mathcal{H}}$ on $\mathcal{H}$ [3].

Let a physical device be described by the measurable map $F: \mathcal{H} \to \mathcal{H}'$, $\mathcal{H}$ and $\mathcal{H}'$ real separable Hilbert spaces. It is of primary importance to be able to compute

$$\mu_G(\omega : F(\omega) \in B), \qquad B \in \mathcal{B}_{\mathcal{H}'},$$

where $\mathscr{B}_{\mathscr{H}'}$ is the Borel σ-algebra on $\mathscr{H}'$. This computation in general is not possible because $\{\omega : F(\omega) \in B\}$ in general does not belong to $\mathscr{C}$. On the other hand the family of sets

$$\mathscr{F}^F = \{F^{-1}(B), B \in \mathscr{B}_{\mathscr{H}'}\}$$

is a sub-σ-algebra of $\mathscr{B}_{\mathscr{H}}$ so that, for some measurable F, μ_G can be extended to $\mathscr{F}^F$. In this case we can qualify F to be a μ_G-random variable (R.V.) and $\mu_G \circ F^{-1}$ is countably additive on $(\mathscr{H}', \mathscr{B}_{\mathscr{H}'})$.

Any measurable function of type $F \circ P_n$, P_n being a finite-dimensional projector, is obviously a μ_G-R.V. Such a class of R.V.s is rather poor for applications; an important larger class of μ_G-R.V. is the class of the so-called physical random variables (P.R.V.) introduced by Balakrishnan [3].

Let $\mathscr{P}(\mathscr{H})$ denote the set of all sequences of finite-dimensional projectors on $\mathscr{H}$ increasing strongly to the identity. We recall the following definition of a physical random variable [3, 4].

Definition 1. Let $(\mathscr{H}, \mathscr{C}, \mu_G)$ be a cylinder-probability triple on the real separable Hilbert space $\mathscr{H}$ endowed with the finite additive cylinder measure μ_G. Then the function $F: (\mathscr{H}, \mathscr{C}, \mu_G) \to \mathscr{B}$, $\mathscr{B}$ a real separable Banach space, is a physical random variable (P.R.V.) if

(i) $\{F \circ P_n\}$ is Cauchy in μ_G-measure for every sequence $\{P_n\} \in \mathscr{P}(\mathscr{H})$, and

(ii) the sequence $\{\nu_n\}$ of probability measures on $\mathscr{B}$ defined by $\nu_n = \mu_G \circ (F \circ P_n)^{-1}$ converges strongly to the same probability measure ν on $\mathscr{B}$ for every sequence $\{P_n\} \in \mathscr{P}(\mathscr{H})$.

When $\mathscr{B}$ is a Hilbert space, condition (ii) is equivalent to saying that the limit

$$C(h) = \lim_{n \to \infty} \int_{\mathscr{H}} e^{i[F \circ P_n \omega, h]} d\mu_G(\omega)$$

is independent of the sequence $\{P_n\}$ chosen.

In the P.R.V. class we can always extend μ_G to events in $\mathscr{F}^F$ by

$$\mu_G \circ F^{-1}(B) = \lim_{n \to \infty} \mu_G(\omega \in \mathscr{H} : F \circ P_n \omega \in B), \qquad B \in \mathscr{B}_{\mathscr{B}}.$$

The class of all $\mathscr{B}$-valued P.R.V.'s is denoted by $\mathscr{L}'(\mathscr{H}, \mathscr{C}, \mu_G; \mathscr{B})$. We need to introduce the following continuity concepts in order to characterize an important subclass of $\mathscr{L}'(\mathscr{H}, \mathscr{C}, \mu_G; \mathscr{B})$.

Definition 2. A map $F: \mathscr{H} \to \mathscr{B}$ is continuous in $x \in \mathscr{H}$ with respect to the S-topology (Sazonov topology) if, for any $\varepsilon > 0$, there exists an H.S. operator $L_\varepsilon(x): \mathscr{H} \to \mathscr{H}$ such that

(1) $$\|L_\varepsilon(x)(x - x')\| < 1$$

implies

(2) $$\|F(x) - F(x')\| < \varepsilon.$$

F is uniformly S-continuous on $U \subset \mathscr{H}$ when the H.S. operator in (1) does not depend on $x \in U$.

Definition 3. A map $F\colon \mathscr{H} \to \mathscr{B}$ is said to be uniformly S-continuous around the origin (U.S.C.A.O.) if F is uniformly S-continuous on the sets

$$U_n = \{x \in \mathscr{H} : \|A_n x\| \le 1\},$$

where $\{A_n\}_1^\infty$ is a sequence of H.S. operators such that

$$\|A_n\|_{\text{H.S.}} \to 0 \quad \text{and} \quad \bigcup_{n=1}^{\infty} U_n = \mathscr{H}.$$

Obviously a map which is uniformly S-continuous on $\mathscr{H}$ is also uniformly S-continuous around the origin. A necessary and sufficient condition for a function F to be U.S.C.A.O. is the existence of an H.S. operator L such that F is uniformly S-continuous on the set $U_m = \{x \in \mathscr{H} : \|Lx\| \le m\}$ for each integer $m > 0$ (see [5, p. 109]). Moreover, the U.S.C.A.O. class is closed with respect to left concatenation with continuous maps [6, 2].

The importance of definitions (2) and (3) is justified by the following theorem, whose proof is essentially given in [6, 7] (see [8] for a simpler proof).

Theorem 1.1. *Let* $F\colon (\mathscr{H}, \mathscr{C}, \mu_G) \to \mathscr{B}$ *be a U.S.C.A.O. map. Then* F *is a P.R.V.*

When $\mathscr{B}$ is a Hilbert space, note that if F is an H.S. operator then F obviously defines a linear P.R.V.

2. The approximation theorem

Let $\mathscr{H}$ and $\mathscr{B}$ be a real separable Hilbert space and a real separable Banach space respectively. Let us recall that an n-linear operator M on $\mathscr{H} \times \cdots \times \mathscr{H}$ (n times) into $\mathscr{B}$ is a map which is linear in each of its arguments separately. It is natural to define

$$M_n x^n = M(x, \ldots, x)$$

as the n-degree monomial operator (associated to the multilinear operator M) on $\mathscr{H}$ into $\mathscr{B}$. Then the operator $P\colon \mathscr{H} \to \mathscr{B}$, defined as

$$P(x) = M_n x^n + M_{n-1} x^{n-1} + \cdots + M_1 x + M_0,$$

where M_0 stands for a constant operator, is called an n-degree polynomial operator. The following generalization of the classical Weierstrass theorem holds [10, 11]:

Theorem 2.1. *The family of continuous polynomials on* $\mathscr{H}$ *into* $\mathscr{B}$, *restricted to a compact set* K *of* $\mathscr{H}$, *is dense in the set of continuous functions on* $\mathscr{H}$ *into* $\mathscr{B}$ *restricted to* K.

We will prove that the physical random variables belonging to the U.S.C.A.O. class can be approximated by polynomials. In order to get this result we need the following representation theorem (first appeared in [12]):

Theorem 2.2. *The map* $F\colon \mathscr{H} \to \mathscr{B}$ *is uniformly S-continuous around the origin if and only if there exists an H.S. operator* $\Lambda\colon \mathscr{H} \to \mathscr{H}$ *and a continuous map* $g\colon \mathscr{H} \to \mathscr{B}$ *such that* $F = g \circ \Lambda$.

Proof. In §1 we saw that a map that is left concatenation of a continuous map with an H.S. operator is in the U.S.C.A.O. class. Only the necessity needs a

proof. Without loss of generality, let us define the set

$$U_m = \{x \in \mathscr{H} : \|\tilde{\Lambda}x\| \le m\}, \qquad m = 1, 2, \ldots,$$

where $\tilde{\Lambda}$ is a suitable H.S. operator such that F is uniformly S-continuous on U_m as established in [5]. Therefore, given any x, y in U_m, it follows that

$$\text{(3)} \qquad \|F(x) - F(y)\| < \tfrac{1}{n}$$

if, for a suitable H.S. operator $L_{n,m}$, we have

$$\text{(4)} \qquad \|L_{n,m}(x - y)\|^2 < 1,$$

or equivalently if

$$\text{(5)} \qquad \left[\frac{1}{2^{n+m+1}} \frac{L_{n,m}^* L_{n,m}}{\|L_{n,m}\|_{\text{H.S.}}^2}(x - y),\ x - y \right] < \frac{1}{2^{n+m+1}\|L_{n,m}\|_{\text{H.S.}}^2} = \delta_{n,m}.$$

Let us introduce the linear operator $R: \mathscr{H} \to \mathscr{H}$

$$R = \sum_{n,m} \frac{1}{2^{n+m+1}} \left(\frac{L_{n,m}^* L_{n,m}}{\|L_{n,m}\|_{\text{H.S.}}^2} + \frac{1}{n}\pi_{\mathscr{H}}^n \right)$$

where $\{\pi_{\mathscr{H}}^n\}$ is a complete sequence of finite-dimensional orthoprojectors such that $\text{rank}(\pi_{\mathscr{H}}^n) = n$, converging strongly to the identity. The operator R is positive definite, selfadjoint, and trace-class since, denoting by $\{e_i\}$ a complete orthonormal system for $\{\pi_{\mathscr{H}}^n\}$, we have

$$\sum_{i=1}^{\infty}[Re_i, e_i] = \sum_{i=1}^{\infty} \left[\sum_{n,m} \frac{1}{2^{m+n+1}} \left(\frac{1}{\|L_{n,m}\|_{\text{H.S.}}^2} L_{n,m}^* L_{n,m} + \frac{1}{n}\pi_{\mathscr{H}}^n \right) e_i,\ e_i \right]$$

$$= \sum_{n,m} \frac{1}{2^{n+m+1}} \sum_{i=1}^{\infty} \left(\frac{1}{\|L_{n,m}\|_{\text{H.S.}}^2}[L_{n,m}e_i,\ L_{n,m}e_i] + \frac{1}{n}[\pi_{\mathscr{H}}^n e_i,\ e_i] \right)$$

$$= 1.$$

Inequality (3), in view of (5), is also implied by

$$\text{(6)} \qquad [R(x - y),\ x - y] < \delta_{n,m},$$

since, as can readily be recognized, it is

$$\left[\frac{1}{2^{n+m+1}} \frac{L_{n,m}^* L_{n,m}}{\|L_{n,m}\|_{\text{H.S.}}^2}(x - y),\ x - y \right] \le [R(x - y),\ x - y].$$

Now consider the factorization

$$\text{(7)} \qquad \Lambda^*\Lambda = R + \tilde{\Lambda}^*\tilde{\Lambda}$$

where Λ is an H.S. full-rank operator, as can easily be verified by noting that at least $\{\|\frac{1}{n}\pi_{\mathscr{H}}^n x\|\}$ is a separable sequence of seminorms.

The inverse operator Λ^{-1} is well defined, and its domain $\mathscr{D}(\Lambda^{-1})$ is dense in $\mathscr{H}$ because of the full-rank property of Λ. Moreover, let us introduce the sequence of sets

$$\overline{U}_m = \{x : \|\Lambda x\| \le m\}.$$

By taking into account (7) and the positive definiteness of R, we have

$$\overline{U}_m \subset U_m.$$

Let us consider the map $\hat{g} \colon \mathscr{D}(\Lambda^{-1}) \to \mathscr{B}$ defined as $\hat{g}(x) = F \circ \Lambda^{-1}x$. The map $\hat{g}$ is uniformly continuous on the sets G_m defined as

$$G_m = \{x \in \mathscr{D}(\Lambda^{-1}), \ \|x\| \le m\}.$$

Note first that if $x \in G_m$ then $x = \Lambda\Lambda^{-1}x$ and $\|\Lambda(\Lambda^{-1}x)\| \le m$ so that $\Lambda^{-1}x \in \overline{U}_m \subset U_m$. Recalling that (6) implies (3) in U_m, for any $x, y \in G_m$, it is

$$\|\hat{g}(x) - \hat{g}(y)\| = \|F \circ \Lambda^{-1}x - F \circ \Lambda^{-1}y\| < 1/n$$

if

(8) $$[R(\Lambda^{-1}x - \Lambda^{-1}y), \ \Lambda^{-1}x - \Lambda^{-1}y] < \delta_{n,m}.$$

From (7) it is $\Lambda^{*}\Lambda \ge R$, so that (8) is implied by

$$\|\Lambda(\Lambda^{-1}x - \Lambda^{-1}y)\|^2 = \|x - y\|^2 < \delta_{n,m},$$

which proves the uniform continuity of $\hat{g}$ on G_m. Finally, denoting by $g \colon \mathscr{H} \to \mathscr{B}$ the unique continuous extension of $\hat{g}$ on the whole Hilbert space $\mathscr{H}$, it results that $F = g \circ \Lambda$, which proves the theorem. □

Remark. Note that the function g is uniformly continuous on any closed sphere. There is no contradiction with the sufficient part of the theorem in that any H.S. operator L can be decomposed in the product of a compact operator C with an H.S. operator Λ [7, Lemma 3.6.1]. Thus, if f is a continuous map, $f \circ L = f \circ C \circ \Lambda$, and the map $g = f \circ C$ is clearly uniformly continuous on the closed spheres.

Now we can state the following approximation theorem.

Theorem 2.3. *Let* $F \colon (\mathscr{H}, \mathscr{C}, \mu_{\mathrm{G}}) \to \mathscr{B}$ *be a P.R.V. belonging to the U.S.C.A.O. class. Then, for each* $\varepsilon > 0$, *there exists a polynomial P.R.V.* P_ε *such that*

$$\mu_{\mathrm{G}}(x \in \mathscr{H} : \|P_\varepsilon(x) - F(x)\| > \varepsilon) < \varepsilon.$$

Proof. By Theorem 2.2 there exists an H.S. operator Λ and a continuous function g such that $F = g \circ \Lambda$. Given $\varepsilon > 0$, by a well-known Prohorov theorem [9] there exists a compact set C_ε in $\mathscr{H}$ such that

$$\mu_{\mathrm{G}} \circ \Lambda^{-1}(C_\varepsilon) > 1 - \varepsilon,$$

since $\mu_{\mathrm{G}} \circ \Lambda^{-1}$ is obviously countably additive. Moreover, by the Weierstrass Theorem 2.1, there exists an n_ε-degree polynomial operator $p^{(n_\varepsilon)}$ such that on the compact set C_ε

$$\sup_{x \in C_\varepsilon} \|p^{(n_\varepsilon)}(x) - g(x)\| \le \varepsilon/2.$$

The map $p^{(n_\varepsilon)} \circ \Lambda$, being the composition of a continuous map with an H.S. operator, is a P.R.V. by Theorems 1.1 and 2.2, and obviously it is a polynomial too. We have

$$\mu_{\mathrm{G}}(\omega : \|p^{(n_\varepsilon)}(\Lambda\omega) - g \circ \Lambda\omega\| > \varepsilon)$$
$$= \mu_{\mathrm{G}}(\{\omega : \|p^{(n_\varepsilon)}(\Lambda\omega) - g \circ \Lambda\omega\| > \varepsilon, \ \Lambda\omega \in C_\varepsilon\}$$
$$\cup \{\omega : \|p^{(n_\varepsilon)}(\Lambda\omega) - g \circ \Lambda\omega\| > \varepsilon, \ \Lambda\omega \notin C_\varepsilon\})$$
$$\le \mu_{\mathrm{G}}(\omega : \Lambda\omega \notin C_\varepsilon) < \varepsilon,$$

so that the P.R.V. $P_\varepsilon = p^{(n_\varepsilon)} \circ \Lambda$ approximates F in the μ_{G}-measure. □

3. Concluding remarks

Theorem 2.2 allows us to recognize that the property of being a physical random variable owned by nonlinear transformations uniformly S-continuous around the origin depends essentially on the Hilbert-Schmidt character of a linear transformation. The decomposition stated by this theorem allows, moreover, a simple characterization of an important class of P.R.V.s as polynomial maps. These results can be a useful tool in modeling stochastic systems and in filtering applications [13]. We recall here that the P.R.V. class considered in Theorem 2.3 contains the Radon-Nikodým derivative with respect to the gaussian measure of the measure generated by the output of a stochastic system [12].

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