PARABOLICITY OF A CLASS OF HIGHER-ORDER ABSTRACT DIFFERENTIAL EQUATIONS

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Abstract. Let $E$ be a complex Banach space, $c_i \in \mathbb{C}$ ($1 \leq i \leq n - 1$), and $A$ be a nonnegative operator in $E$. We discuss the parabolicity of the higher-order abstract differential equations

$$(*) \quad u^{(n)}(t) + \sum_{i=1}^{n-1} c_i A^i u^{(n-i)}(t) + Au(t) = 0$$

and some perturbation cases of $(*)$. A sufficient and necessary condition for $(*)$ to be parabolic is obtained, provided $k_1 > k_2 - k_1 > \ldots > 1 - k_{n-1} > 0$, $c_i \neq 0$ ($1 \leq i \leq n - 1$). For $A$ strictly nonnegative (Definition 1.3), $n = 3$, $c_1, c_2 \geq 0$, a sharp criterion is given.

1. Introduction and preliminaries

Throughout this paper, $E$ will be a complex Banach space and $n \in \mathbb{N}$ (the set of natural numbers). For $\theta \in (0, \pi/2]$ and $\omega \in \mathbb{R}$ (the set of real numbers), write

$$\sum(\theta, \omega) = \{z \in \mathbb{C}: z \neq \omega, |\arg(z - \omega)| < \frac{\pi}{2} + \theta\},$$

$$\sum_\theta = \{z \in \mathbb{C}: z \neq 0, |\arg z| < \theta\}.$$

Definition 1.1. Suppose $A_1, \ldots, A_n$ are closed linear operators in $E$ and $\theta \in (0, \pi/2]$. We say $[A_1, \ldots, A_n] \in \mathcal{A}_0(\theta)$, if for each $\theta' \in (0, \theta)$ there exist $C_{\theta'}$, $\omega_{\theta'} > 0$ such that

$$(1.1) \quad \left\| \lambda^{n-i} A_i \left( \lambda^n + \sum_{i=1}^n \lambda^{n-i} A_i \right)^{-1} \right\| \leq C_{\theta'},$$

whenever $\lambda \in \sum(\theta', \omega_{\theta'})$, $1 \leq i \leq n$.

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Write $\mathcal{A}_n = \bigcup_{\theta \in (0, \pi/2]} \mathcal{A}_n(\theta)$. When $[A_1, \ldots, A_n] \in \mathcal{A}_n$, we also say the abstract differential equation

\begin{equation}
(1.2) \quad u^{(n)}(t) + \sum_{i=1}^{n} A_i u^{(n-i)}(t) = 0
\end{equation}

is parabolic.

Clearly, when $[A_1, \ldots, A_n] \in \mathcal{A}_n(\theta)$ ($\theta \in (0, \pi/2]$), (1.1) also holds for $i = 0$ ($A_0 = I$, the identity operator); $[A_1] \in \mathcal{A}_1(\theta)$ for some $\theta \in (0, \pi/2]$ iff $-A_1$ is the generator of an exponentially bounded holomorphic semigroup.

We note that parabolicity of equation (1.2) is 'comparable' with existence of an analytic exponentially bounded semigroup for the corresponding first-order system

\begin{equation}
(1.3) \quad v'(t) + G_n v(t) = 0,
\end{equation}

in a proper $B$-space, where $v = (u_0, u_1, \ldots, u_{n-1}) = (u, u', \ldots, u^{(n-1)})$ and

\begin{equation}
G_n = \begin{pmatrix}
0 & -I & 0 & \cdots & 0 \\
0 & 0 & -I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -I \\
A_n & A_{n-1} & A_{n-2} & \cdots & A_1
\end{pmatrix}.
\end{equation}

It is known that the parabolicity of (1.2) ensures the existence and uniqueness of the solution of the Cauchy problem for (1.2) (see, e.g., [10, 11]). But under what condition is equation (1.2) parabolic? For second-order equations, this problem has been studied by many authors (see, e.g., [1-3, 5-9] and references therein); here (1.2) amounts to

\begin{equation}
(1.5) \quad u''(t) + A_1 u'(t) + A_2 u(t) = 0,
\end{equation}

and (1.3) to

\begin{equation}
(1.6) \quad G_2 = \begin{pmatrix}
0 & -I \\
A_2 & A_1
\end{pmatrix}.
\end{equation}

In [1] Chen and Russell posed two conjectures in their study of linear elastic systems with structure damping, which state, qualitatively, that $-G_2$ (in a suitable product space) generates an exponentially bounded analytic semigroup in the case where $A_1, A_2$ are positive and selfadjoint operators in a Hilbert space and the dissipation operator $A_1$ is 'comparable' with the $\frac{1}{2}$th power of the elastic operator $A_2$. Huang [6, 7] and, independently, Chen and Triggiani [2] proved these two conjectures; furthermore, they [3, 8, 9] discussed the general case where $A_1$ is 'comparable' with the $\alpha$th power of $A_2$ over the entire range $0 \leq \alpha \leq 1$ of the parameter $\alpha$. Recently, in the framework of Banach spaces, Favini and Obrecht [10] studied sufficient and necessary conditions ensuring equation (1.5) with $A_1 = pA_2^\alpha$ ($p \in \mathbb{C}$, $0 < \alpha < 1$) parabolic.

This paper aims at investigating the parabolicity of (1.2) for any $n$, but under the special condition $A_n = A > 0$, $A_i = c_i A_i$ ($c_i \in \mathbb{C}$, $1 \leq i \leq n - 1$), that is,

\begin{equation}
(1.7) \quad u^{(n)}(t) + \sum_{i=1}^{n-1} c_i A_i u^{(n-i)}(t) + A u(t) = 0.
\end{equation}
First (in §2), assuming \( c_i \neq 0 \) \((1 \leq i \leq n - 1)\), \( k_1 > k_2 - k_1 > \cdots > 1 - k_{n-1} > 0 \), we obtain a sufficient and necessary condition for (1.7) to be parabolic. Furthermore, some perturbation theorems are presented. Following this (in §3), we specialize to the case where \( A \) is strictly nonnegative, \( n = 3, c_1, c_2 \geq 0 \), and give a complete and clear answer for the problem of whether (1.7) is parabolic.

**Definition 1.2.** Suppose \( S \) is an arbitrary linear operator in \( E \). \( S \) is nonnegative, if for each \( \lambda > 0, \lambda \in \rho(-S) \) and

\[
\sup\{\|\lambda(\lambda + S)^{-1}\|: \lambda > 0\} < +\infty.
\]

It can be shown (cf. [4, Lemma 6.4.1]) that, if \( S \) is a nonnegative operator in \( E \), then there exists \( \theta \in (0, \pi/2) \) such that \( \lambda \in \rho(-S) \) for each \( \lambda \in \Sigma_{\theta} \) with \( \{\|\lambda(\lambda + S)^{-1}\|: \lambda \in \Sigma_{\theta}\} \) bounded.

Let \( S \) be a nonnegative operator in \( E \). Set as in [5]

\[
\theta^+(S) = \inf\{\theta \in (-\pi, \pi): \text{there exist } C, \omega > 0 \text{ such that, for each } \lambda \text{ with } |\lambda| \geq \omega \text{ and } \theta \leq \arg \lambda \leq \pi, \lambda \in \rho(S) \text{ and } \|\lambda(\lambda + S)^{-1}\| \leq C\},
\]

\[
\theta^-(S) = \sup\{\theta \in (-\pi, \pi): \text{there exist } C, \omega > 0 \text{ such that, for each } \lambda \text{ with } |\lambda| \geq \omega \text{ and } -\pi \leq \arg \lambda \leq \theta, \lambda \in \rho(S) \text{ and } \|\lambda(\lambda + S)^{-1}\| \leq C\}.
\]

Obviously, \( \theta^+_\infty(S) \geq \theta^-\infty(S) \); \([S] \in A_1(\theta) \ (\theta \in (0, \pi/2])\) iff \( \theta^+_\infty(S) \leq \pi/2 - \theta \) and \( \theta^-_\infty(S) \geq -\pi/2 + \theta \). It is not difficult to verify that, for \( c \in \mathbb{C}, cS \) is nonnegative iff either

(i) \( \arg c < -\pi - \theta^+_\infty(S) \), or
(ii) \( -\pi - \theta^-_\infty(S) < \arg c < \pi - \theta^+_\infty(S) \), or
(iii) \( \arg c > \pi - \theta^-_\infty(S) \),

and if \( cS \) is nonnegative, we have

\[
\theta^\pm(cS) = \begin{cases} 
\arg c + \theta^\pm(S) + 2\pi & \text{if } \arg c < -\pi - \theta^+_\infty(S), \\
\arg c + \theta^\pm(S) & \text{if } -\pi - \theta^-_\infty(S) < \arg c < \pi - \theta^+_\infty(S), \\
\arg c + \theta^\pm(S) + 2\pi & \text{if } \arg c > \pi - \theta^-_\infty(S). 
\end{cases}
\]

Finally, for each \( 0 < \alpha < 1 \), as pointed out in the proof of [5, Lemma 3.3],

\[
\theta^\pm(S^\alpha) = \alpha \theta^\pm(S).
\]

**Definition 1.3.** We say that \( S \) is strictly nonnegative if \( \theta\infty(S) = 0 \).

2. **Results for Arbitrary Order**

Throughout this section, \( A \) will be a densely defined and nonnegative operator in \( E \), \( c_i \in \mathbb{C} \) \((1 \leq i \leq n - 1)\), and

\[
P_0(\lambda) = \lambda^n + \sum_{i=1}^{n-1} c_i A^k \lambda^{n-i} + A.
\]

First, we state the well-known Moment inequality:
Let $0 \leq \alpha < \beta < \varepsilon \leq 1$. Then there exists a constant $C = C(\alpha, \beta, \varepsilon)$ such that

$$
(2.1) \quad \|A^\beta u\| \leq C\|A^\alpha u\|^{(\beta - \alpha)/(\varepsilon - \alpha)} \|A^\varepsilon u\|^{(\varepsilon - \beta)/(\varepsilon - \alpha)} \quad (u \in D(A^\varepsilon)).
$$

**Theorem 2.1.** Let $k_1 > k_2 - k_1 > \cdots > k_{n-1} - k_{n-2} > 1 - k_{n-1} > 0$, $c_i \neq 0$ for each $1 \leq i \leq n - 1$. Then $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$ ($\theta \in (0, \pi/2]$) iff, for each $1 \leq i \leq n$, $[c_i^{-1} c_i A^{k_i - k_{i-1}}] \in \mathcal{A}_i(\theta)$, where $c_0 = c_n = 1$.

**Proof. Sufficiency.** Set $c_i^{-1} c_i = \tilde{c}_i$, $k_i - k_{i-1} = t_i$ ($1 \leq i \leq n$),

$$
P_i(\lambda) = \prod_{i=1}^n (\lambda + \tilde{c}_i A^{t_i}),
$$

$$
Q(\lambda) = \sum_{m=1}^{n-1} \sum_{(i_1, \ldots, i_m) \in I_m} \tilde{c}_{i_1} \cdots \tilde{c}_{i_m} A^{t_{i_1} + \cdots + t_{i_m}} \lambda^{n-m},
$$

where, for each $1 \leq m \leq n - 1$,

$$
I_m = \{(i_1, \ldots, i_m) : 1 \leq i_1 < \cdots < i_m \leq n, (i_1, \ldots, i_m) = (1, \ldots, m)\}.
$$

Then

$$
t_1 > t_2 > \cdots > t_n, \quad \lambda_m = \sum_{i=1}^{m} t_i \quad (1 \leq m \leq n - 1).
$$

By hypothesis, for each $\theta' \in (0, \theta)$, there exist $C_{\theta'}$, $\omega_{\theta'} > 0$ such that

$$
(2.2) \quad \|\lambda^{n-m} A^{t_1 + \cdots + t_m} P^{-1}_m(\lambda)\| \leq C_{\theta'}, \quad \|\lambda^m P^{-1}_m(\lambda)\| \leq C_{\theta'}
$$

whenever $\lambda \in \sum(\theta', \omega_{\theta'})$, $1 \leq i_1 < \cdots < i_m \leq n$, $1 \leq m \leq n$. This together with (2.1) yields that, for each $\theta' \in (0, \theta)$, there exist $C$, $\tilde{C}_{\theta'}$, $\omega_{\theta'} > 0$ such that, for $\lambda \in \sum(\theta', \omega_{\theta'})$, $(i_1, \ldots, i_m) \in I_m$, $1 \leq m \leq n - 1$,

$$
\|\lambda^{n-m} A^{t_1 + \cdots + t_m} P^{-1}_m(\lambda)\| \leq C_{\theta'}\|\lambda^{n-m} A^{k_1 - k_0} P^{-1}_m(\lambda)\| \leq C(\tilde{C}_{\theta'})\|\lambda^{n-m} A^{k_1 - k_0} P^{-1}_m(\lambda)\| \leq C(\tilde{C}_{\theta'}) \|\lambda^{n-m} A^{k_1 - k_0} P^{-1}_m(\lambda)\| \leq 2c_m C_{\theta'},
$$

which approaches 0 as $|\lambda| \to \infty$.

Therefore, for each $\theta' \in (0, \theta)$, there is $\tilde{\omega}_{\theta'} > \omega_{\theta'}$ such that for $\lambda \in \sum(\theta', \tilde{\omega}_{\theta'})$

$$
(2.3) \quad \|Q(\lambda) P^{-1}_m(\lambda)\| < \frac{1}{2}.
$$

Thus using (2.2) again we obtain that, for each $\theta' \in (0, \theta)$, $\lambda \in \sum(\theta', \tilde{\omega}_{\theta'})$, $1 \leq m \leq n$,

$$
\|c_m\lambda^{n-m} A^{k_1} P^{-1}_m(\lambda)\| \leq \|c_m\lambda^{n-m} A^{k_1} P^{-1}_m(\lambda)[I - Q(\lambda) P^{-1}_m(\lambda)]^{-1}\| \leq 2c_m C_{\theta'}.
$$

In conclusion, $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$.

**Necessity.** Making use of (2.1) as in the proof of sufficiency, we obtain that, for each $\theta' \in (0, \theta)$, there exists $\omega_{\theta'} > 0$ such that, for $\lambda \in \sum(\theta', \omega_{\theta'})$, $(i_1, \ldots, i_m) \in I_m$, $1 \leq m \leq n$,

$$
(2.4) \quad \|\lambda^{n-m} A^{t_1 + \cdots + t_m} P^{-1}_m(\lambda)\| \leq C_{\theta'}\|\lambda^{(t_1 + \cdots + t_m)k_{m-1} - 1} A\|^m.
$$
and therefore there exist $\hat{\omega}_{\theta'} > \omega_{\theta'}$, $M_{\theta'} > 0$ such that, for $\lambda \in \sum(\theta', \hat{\omega}_{\theta'})$, 

$$
\left\{ \begin{array}{l}
\|Q(\lambda)P_0^{-1}(\lambda)\| < \frac{1}{2}, \\
\|\lambda(\lambda + \hat{c}_m A^{l_m})^{-1}P_1(\lambda)P_0^{-1}(\lambda)\| \leq M_{\theta'}.
\end{array} \right.
$$

Accordingly, for each $\theta' \in (0, \theta)$, $\lambda \in \sum(\theta', \omega_{\theta'})$, $1 \leq m \leq n$,

$$
\|\lambda(\lambda + \hat{c}_m A^{l_m})^{-1}\| = \|\lambda(\lambda + \hat{c}_m A^{l_m})^{-1}P_1(\lambda)P_0^{-1}(\lambda)[I + Q(\lambda)P_0^{-1}(\lambda)]\| \leq 2M_{\theta'}.
$$

This ends the proof. Q.E.D.

Corollary 2.2. Let $\theta_+^\infty(A) = 0$, $c_{j} > 0$ ($1 \leq i \leq n - 1$), and $k_1 > k_2 - k_1 > \ldots > k_{n-1} - k_{n-2} > 1 - k_{n-1} > 0$. Then $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\pi/2)$.

The following are some perturbation cases.

**Theorem 2.3.** Let $B_1, \ldots, B_{n-1}$ be closed linear operators in $E$ satisfying that, for each $1 \leq m \leq n - 1$, there is $l_m$ with $k_{m-1} < l_m < \frac{1}{2}(k_{m-1} + k_{m+1})$ such that $D(B_m) \supset D(A^{l_m})$. Then if $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$ ($\theta \in (0, \pi/2]$), so does $[c_1 A^{k_1} + B_1, \ldots, c_{n-1} A^{k_{n-1}} + B_{n-1}, A]$.

**Proof.** By hypothesis, there is $C > 0$ such that, for each $1 \leq m \leq n - 1$, 

$$
\|u\| \leq C\|x\| + C\|A^{l_m}u\|.
$$

So using (2.1) yields that, for each $\theta' \in (0, \theta)$, there exist $C_{\theta'}$, $\omega_{\theta'} > 0$ such that, for each $1 \leq m \leq n - 1$, $\lambda \in \sum(\theta', \omega_{\theta'})$,

$$
\|\lambda^{-m} B_m P_0^{-1}(\lambda)\| \leq C|\lambda|^{-m} \|P_0^{-1}(\lambda)\| + C|\lambda|^{-m} \|A^{l_m} P_0^{-1}(\lambda)\|
$$

\[
\leq CC_{\theta'}|\lambda|^{-m} + C|\lambda|^{-m} \|A^{k_{m-1}} P_0^{-1}(\lambda)\|\|\|A^{k_{m+1}} P_0^{-1}(\lambda)\|^{1-\tau}
\]

\[
= CC_{\theta'}(|\lambda|^{-m} + |\lambda|^{-(m-n-1)\tau}|\lambda|^{(m-n+1)(1-\tau)})
\]

which approaches 0 as $|\lambda| \to \infty$, where $\tau = (l_m - k_{m-1})(k_{m+1} - k_{m-1})^{-1} < 1$.

Consequently, for each $\theta' \in (0, \theta)$ there is $\hat{\omega}_{\theta'} > \omega_{\theta'}$ such that, for $\lambda \in \sum(\theta', \hat{\omega}_{\theta'})$,

$$
\left\| \sum_{m=1}^{n-1} \lambda^{-m} B_m P_0^{-1}(\lambda) \right\| < \frac{1}{2}.
$$

This leads to the result as claimed. Q.E.D.

**Corollary 2.4.** Let $0 < k_1 < \ldots < k_{n-1} < 1$ and $k_j < \frac{1}{2}(k_{j-1} + k_{j+1})$ for some $1 \leq j \leq n - 1$. Then $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$ ($\theta \in (0, \pi/2]$) implies $[c_1 A^{k_1}, \ldots, c_{j-1} A^{k_{j-1}}, 0, c_{j+1} A^{k_{j+1}}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$.

**Theorem 2.5.** Let $\hat{c}_i \in \mathbb{C}$, $0 < l_i < i/n$ for each $1 \leq i \leq n - 1$, $\theta \in (0, \pi/2]$. Then $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$ iff $[c_1 A^{k_1} + \hat{c}_1 A^{l_1}, \ldots, c_{n-1} A^{k_{n-1}} + \hat{c}_{n-1} A^{l_{n-1}}, A] \in \mathcal{A}_n(\theta)$.

This theorem is an immediate consequence of the following (general) result by taking $i_m = n$, $\epsilon_m = l_m$ ($1 \leq m \leq n - 1$), and using (2.1).
Theorem 2.6. Assume \([A_1, \ldots, A_n] \in \mathscr{S}_r(\theta)\) for some \(\theta \in (0, \pi/2]\), and \(B_m, A_m + B_m\) are closed and densely defined linear operators in \(E\) \((1 \leq m \leq n)\). If, for each \(1 \leq m \leq n\), there exist \(i_m, \varepsilon_m\) with \(1 \leq i_m \leq n\), \(0 < \varepsilon_m \leq 1\) such that \(D(B_m) \supset D(A_{i_m})\) and, for each \(u \in D(A_{i_m})\),
\[
\|B_m u\| \leq C\|u\| + C\|A_{i_m} u\|^{\varepsilon_m} \|u\|^{1-\varepsilon_m}\text{ for some }C > 0,
\]
then, for each \(\theta' \in (0, \theta)\), there is \(\omega_{\theta'} > 0\) such that, for \(1 \leq m \leq n\),
\[
\sup_{\lambda \in \Sigma(\theta', \omega_{\theta'})} \|\lambda^{n-m} B_m P^{-1}(\lambda)\| < \begin{cases} \frac{1}{2} & \text{if } i_m \varepsilon_m < m, \\ +\infty & \text{if } i_m \varepsilon_m = m, \end{cases}
\]
where \(P(\lambda) = \lambda^n + \sum_{i=1}^n \lambda^{n-i} A_i\). Furthermore, when \(i_m \varepsilon_m < m\), \([A_1 + B_1, \ldots, A_n + B_n] \in \mathscr{A}_r(\theta)\).

Proof. Observing that, for each \(\theta' \in (0, \theta)\), there exist \(C_{\theta'}, \omega_{\theta'} > 0\) such that, for \(\lambda \in \Sigma(\theta', \omega_{\theta'}), 1 \leq m \leq n\),
\[
\|\lambda^{n-m} B_m P^{-1}(\lambda)\| \\
\leq C\|\lambda^{n-m} P^{-1}(\lambda)\| + C\|\lambda^{n-m} A_{i_m} P^{-1}(\lambda)\|^{\varepsilon_m} \|P^{-1}(\lambda)\|^{1-\varepsilon_m} \leq C C_{\theta'} (\|\lambda|^{-m} + |\lambda|^{n-m} |A_{i_m}|^{\varepsilon_m} |\lambda|^{n(1-\varepsilon_m)}) \leq C C_{\theta'} (|\lambda|^{-m} + |\lambda|^{i_m \varepsilon_m - m}),
\]
we obtain (2.5). The remaining part follows from the plain equality
\[
\left(P(\lambda) + \sum_{m=1}^n \lambda^{n-m} B_m\right)^{-1} = P^{-1}(\lambda) \left[I + \sum_{m=1}^n \lambda^{n-m} B_m P^{-1}(\lambda)\right]^{-1}. \text{ Q.E.D.}
\]

3. The case of \(n = 3\)

Throughout this section, \(A\) will be densely defined, unbounded, and strictly nonnegative. First we state several basic facts.

Basic Facts. For \(0 < \beta \leq 1\), \(a > 0\), \(\text{Re } c > 0\), we have:
(i) \(\theta_\infty^\pm (aA^\beta) = 0\);
(ii) \([cA^\beta] \in \mathscr{A}, [-cA^\beta] \notin \mathscr{A}\);
(iii) for \(b \in \mathbb{R}\), \([bA^{\beta/2}, aA^\beta] \in \mathscr{A}_2\) iff \(b > 0\);
(iv) (see [5, Theorem 3.7]) let \(\frac{1}{2} < \beta < 1\); then \([cA^\beta, A] \in \mathscr{A}_2\) iff
\[
\begin{cases} \arg c > -\pi/2 + \max\{(1-\beta)\theta_\infty^+(A), -\beta\theta_\infty^-(A)\}, \\ \arg c < \pi/2 - \max\{\beta\theta_\infty^+(A), -(1-\beta)\theta_\infty^-(A)\}. \end{cases}
\]

Theorem 3.1. Let \(a_1, a_2 > 0\) and \(0 < k_1, k_2 < 1\). Then
\([a_1 A^{k_1}, 0, A], [0, a_2 A^{k_2}, A], [0, 0, A] \notin \mathscr{A}_3\).

Proof. Observe that, for each \(y_1 \geq 0\), the function
\[
y(x) = x^{-1} + x(y_1 - x)
\]
is continuous in \((0, +\infty)\), and \(y \to +\infty\) as \(x \to 0^+\), \(y \to -\infty\) as \(x \to +\infty\). Hence, for each \(y_1, y_2 \geq 0\), there exists \(x_1 > 0\) such that
\[
y_2 = x_1^{-1} + x_1(y_1 - x_1).
\]
Set \( x_2 = y_1 - x_1 \). If \( x_2 > 0 \), i.e., \( y_1 > x_1 \), then \( y_2 > x^{-1} \); therefore, \( y_1 y_2 > y_1 x_1^{-1} > 1 \). If \( x_2 \leq 0 \), i.e., \( y_1 \leq x_1 \), then \( y_2 \leq x_1^{-1} \); therefore, \( y_1 y_2 > y_1 x_1^{-1} > 1 \). In other words,

| \( y_1 y_2 > 1 \) | \( x_2 > 0 \) |
| \( y_1 y_2 \leq 1 \) | \( x_2 \leq 0 \) |

So from the equality

\[
\lambda^3 + y_1 A^{1/3} \lambda^2 + y_2 A^{2/3} \lambda + A = (\lambda + x_1 A^{1/3})(\lambda^2 + x_2 A^{1/3} \lambda + x_1^{-1} A^{2/3}),
\]

we see by Basic Facts (ii) and (iii)

\[ (3.1) \quad [y_1 A^{1/3}, y_2 A^{2/3}, A] \in \mathcal{A}_3 \quad \text{if} \quad y_1 y_2 > 1. \]

But

\[ (3.2) \quad [y_1 A^{1/3}, y_2 A^{2/3}, A] \notin \mathcal{A}_3 \quad \text{if} \quad y_1 y_2 \leq 1. \]

In fact, if \([y_1 A^{1/3}, y_2 A^{2/3}, A] \in \mathcal{A}_3 \) \((y_1 y_2 \leq 1)\), then by virtue of (2.5) we have that there are \( C, \omega > 0 \), \( \theta \in (0, \pi/2] \) such that, for \( \lambda \in \sum(\theta, \omega), i = 1, 2, 3, \)

\[
||\lambda^{3-i} A^{i/3}(\lambda^3 + y_1 A^{1/3} \lambda^2 + y_2 A^{2/3} \lambda + A)^{-1}|| \leq C.
\]

According to this, the equality

\[
(\lambda^2 + x_2 A^{1/3} \lambda + x_1^{-1} A^{2/3})^{-1} = (\lambda + x_1 A^{1/3})(\lambda^2 + y_1 A^{1/3} \lambda^2 + y_2 A^{2/3} \lambda + A)^{-1}
\]

shows \([x_2 A^{1/3}, x_1^{-1} A^{2/3}] \in \mathcal{A}_2 \), which contradicts Basic Fact (iii). So (3.2) holds. (3.2) indicates \([a_1 A^{1/3}, 0, A], [0, a_2 A^{2/3}, A], [0, 0, A] \notin \mathcal{A}_3 \). Since \([0, 0, A] \notin \mathcal{A}_3 \), using Theorem 2.5 yields that \([a_1 A^{k_1}, 0, A], [0, a_2 A^{k_2}, A] \notin \mathcal{A}_3 \) if \( k_1 < \frac{1}{3}, k_2 < \frac{2}{3} \). Finally, we have that, for each \( a > 0, 0 < \beta < 1, \)

\[ [a A^\beta, a^{-1} A^{1-\beta}, A] \notin \mathcal{A}_3. \]

Indeed, if not, then the equality

\[
(\lambda^2 + A^{-1} A^{1-\beta})^{-1} = (\lambda + a A^\beta)(\lambda^3 + a A^\beta \lambda^2 + a^{-1} A^{1-\beta} \lambda + A)^{-1}
\]

yields \([0, a^{-1} A^{1-\beta}] \in \mathcal{A}_2 \), which contradicts Basic Facts (iii). Thus, we conclude by Theorem 2.5 again that

\[
[a A^\beta, 0, A] \notin \mathcal{A}_3 \quad \text{if} \quad \beta > \frac{1}{3},
\]

\[
[0, a^{-1} A^{1-\beta}, A] \notin \mathcal{A}_3 \quad \text{if} \quad \beta < \frac{1}{3}.
\]

The proof is then complete. Q.E.D.

**Theorem 3.2.** Let \( a_1, a_2 > 0 \) and \( 0 < k_1 < k_2 < 1 \). Then \([a_1 A^{k_1}, a_2 A^{k_2}, A] \in \mathcal{A}_3 \) iff either

(i) \( k_1 > \frac{1}{3}, \quad \frac{1}{2}(1 + k_1) \leq k_2 \leq 2k_1 \), or

(ii) \( k_1 = \frac{1}{3}, \quad k_2 = \frac{2}{3}, \quad a_1 a_2 > 1. \)

**Proof.** Observing

\[
\lambda^3 + (a_2 a_1^{-1} A^{(1-k_1)/2} + a_1 A^{k_1}) \lambda^2 + (a_1^{-1} A^{1-k_1} + a_2 A^{1+k_1}/2) \lambda + A
\]

\[ = (\lambda + a_1 A^{k_1})(\lambda^2 + a_2 a_1^{-1} A^{(1-k_1)/2} \lambda + a_1^{-1} A^{1-k_1}) ,
\]

we obtain

\[ [a_2 a_1^{-1} A^{(1-k_1)/2} + a_1 A^{k_1}, a_1^{-1} A^{1-k_1} + a_2 A^{1+k_1}/2, A] \in \mathcal{A}_3. \]
Thus appealing to Theorem 2.5 gives

\[(3.3) \quad [a_1 A^{k_1}, a_2 A^{(1+k_1)/2}, A] \in \mathcal{A}_3 \quad (k_1 > \frac{1}{2}).\]

Next, let \( \frac{1}{3} < k_1 < \frac{1}{2}. \) Set \( \tau = k_1 (1 - k_1)^{-1} \);

\[
\begin{align*}
  b_1 &= \begin{cases} 
    \frac{1}{2} [a_1 + (a_1^2 - 4a_2)^{1/2}] & \text{if } a_1^2 \geq 4a_2, \\
    r e^{i\theta} & \text{if } a_1^2 < 4a_2;
  \end{cases} \\
  b_2 &= \begin{cases} 
    \frac{1}{2} [a_1 - (a_1^2 - 4a_2)^{1/2}] & \text{if } a_1^2 \geq 4a_2, \\
    r e^{-i\theta} & \text{if } a_1^2 < 4a_2,
  \end{cases}
\end{align*}
\]

where \( \theta = \arccos(\frac{1}{2}a_1 a_2^{-1/2}), \) \( r = a_2^{1/2}, \)

\[
B = r^{-1} e^{-i\theta} A^{1-k_2}.
\]

Then \( \theta^+\infty(B) = -\theta, \quad \frac{1}{2} < \tau < 1, \quad b_1 + b_2 = a_1, \) and \( b_1 b_2 = a_2. \) Therefore, if \( a_1^2 < 4a_2, \)

\[
\begin{align*}
  \max\{(1 - \tau)\theta^+_\infty(B), -\tau\theta^-\infty(B)\} &= \theta\tau, \\
  \max\{\tau\theta^+_\infty(B), -(1 - \tau)\theta^-\infty(B)\} &= \theta(1 - \tau),
\end{align*}
\]

which implies by Basic Facts (iv) that

\[
[r^{1+e^{(\tau-1)i}}B^\tau, B] \in \mathcal{A}_3.
\]

Consequently, using

\[
\begin{align*}
  \lambda^3 + a_1 A^{k_1} \lambda^2 + (a_2 A^{2k_1} + b_1^{-1} A^{1-k_1}) \lambda + A \\
  = (\lambda + b_1 A^{k_1})(\lambda^2 + b_2 A^{k_1} \lambda + b_1^{-1} A^{1-k_1}), \\
  \lambda^2 + b_2 A^{k_1} \lambda + b_1^{-1} A^{1-k_1} = \lambda^2 + r^{1+e^{(\tau-1)i}}B^\tau \lambda + B \quad \text{if } a_1^2 < 4a_2,
\end{align*}
\]

we see by Basic Facts (ii) and (iii) that

\[
[a_1 A^{k_1}, a_2 A^{2k_1} + r^{-1} e^{-i\theta} A^{1-k_1}, A] \in \mathcal{A}_3.
\]

Since \( 1 - k_1 < \frac{2}{3}, \) we claim using Theorem 2.5 again that

\[(3.4) \quad [a_1 A^{k_1}, a_2 A^{2k_1}, A] \in \mathcal{A}_3, \quad \frac{1}{3} < k_1 < \frac{1}{2}.
\]

In conclusion, Corollary 2.2, combined with (3.1), (3.3), and (3.4), shows the "if part". For the "only if part", apply Theorems 3.1 and 2.5 and see that

\[
[a_1 A^{k_1}, a_2 A^{k_1}, A] \notin \mathcal{A}_3 \quad \text{if } k_1 < \frac{1}{3} \text{ or } k_2 > \frac{3}{2}.
\]

Furthermore, Corollary 2.4, together with Theorem 3.1, gives that

\[
[a_1 A^{k_1}, a_2 A^{k_2}, A] \notin \mathcal{A}_3 \quad \text{if } k_2 < \frac{1}{2}(1 + k_1) \text{ or } k_2 > 2k_1.
\]

Then referring to (3.2) ends the proof. \( \text{Q.E.D.} \)

\textbf{Remark.} If \( a_1, a_2 > 0, k_1 \geq k_2, \) then \( [a_1 A^{k_1}, a_2 A^{k_1}, A] \notin \mathcal{A}_3. \) Indeed by virtue of Theorem 2.6 \( [a_1 A^{k_1}, a_2 A^{k_1}, A] \in \mathcal{A}_3 \) implies \( [a_1 A^{k_1}, 0, A] \in \mathcal{A}_3, \) which contradicts Theorem 3.1. Again by Theorem 2.6, if \( k_2 \geq 1, \) then \( [a_1 A^{k_1}, a_2 A^{k_2}, A] \in \mathcal{A}_3 \) iff \( [a_1 A^{k_1}, a_2 A^{k_2}] \in \mathcal{A}_2. \)

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