TORSION UNITS IN INTEGRAL GROUP RINGS

ANGELA VALENTI

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Abstract. Let $G = \langle a \rangle \rtimes X$ where $\langle a \rangle$ is a cyclic group of order $n$, $X$ is an abelian group of order $m$, and $(n, m) = 1$. We prove that if $\mathbb{Z}G$ is the integral group ring of $G$ and $H$ is a finite group of units of augmentation one of $\mathbb{Z}G$, then there exists a rational unit $\gamma$ such that $H^\gamma \subseteq G$.

Let $G$ be a finite group, $\mathbb{Z}G$ the integral group ring of $G$, and $U_1 \mathbb{Z}G$ the group of units of augmentation one in $\mathbb{Z}G$. It has been conjectured by Zassenhaus that if $H$ is a finite subgroup of $U_1 \mathbb{Z}G$, then $H$ is conjugate to a subgroup of $G$ by a rational unit, i.e., there exists $\gamma \in U\mathbb{Q}G$ such that $H^\gamma \subseteq G$.

This conjecture has been confirmed by Weiss in [8] for $p$-groups. In this note we shall prove this conjecture for a certain class of metabelian groups. More precisely, we will establish the following result:

Theorem. Let $G$ be a split extension $\langle a \rangle \rtimes X$, where $\langle a \rangle$ is a cyclic group of order $n$, $X$ is an abelian group of order $m$, and $(n, m) = 1$. If $H \subseteq U_1 \mathbb{Z}G$ is a finite group, then there exists $\gamma \in U\mathbb{Q}G$ such that $H^\gamma \subseteq G$.

Before proceeding to the proof of the theorem we record some useful facts that will be needed.

If $N$ is a normal subgroup of $G$, let us denote by $\Delta(G, N)$ the kernel of the natural map $\mathbb{Z}G \to \mathbb{Z}(G/N)$. Also we briefly write $u \sim g$ to indicate that $u$ is conjugate in $\mathbb{Q}G$ to $g$.

Lemma 1. Let $G = A \rtimes X$, where $A$ is an abelian normal $p$-group and $X$ is any group with $(|A|, |X|) = 1$. Let $u \in U_1 \mathbb{Z}G$ be a unit of the form $u = vw$, where $v \in U(1 + \Delta(G, A))$, $w \in U_1 \mathbb{Z}X$. If $u$ has finite order not divisible by $p$, then $u \sim w$.

Proof. See [3, Lemma 2].

Lemma 2. Let $G = \langle a \rangle \rtimes X$, where $o(a) = n$, $|X| = m$, and $(n, m) = 1$. If an element $a^r x$ of $G$, where $x \in X$, is of order divisible by all primes dividing $n$, then $x$ is central in $G$.

Proof. It is a consequence of [4, Lemma 2.3].

Lemma 3. Let $G$ be a split extension $\langle a \rangle \rtimes X$, where $\langle a \rangle$ is a cyclic group of order $n$, $X$ is an abelian group of order $m$, and $(n, m) = 1$. If $G_0$ is a...
subgroup of $G$, then $G_0 = \langle b \rangle \times X_0$, where $b \in \langle a \rangle$ and $X_0$ is isomorphic to a subgroup of $X$.

Proof. If $\varphi : G_0 \rightarrow X$ is such that $a^j x \rightarrow x$, then $\varphi$ is a homomorphism and $\ker \varphi = \langle b \rangle$ for some $b \in \langle a \rangle$. Hence $G_0/\ker \varphi \cong \varphi(G_0)$ and $G_0 = \langle b \rangle \times X_0$, with $X_0$ isomorphic to a subgroup of $X$.

The proof of the theorem will be based on the following reduction:

Lemma 4. Let $G$ be a finite group, $G_0$ a subgroup of $G$, and $H$ a finite subgroup of $U_1ZG$. Suppose that there exists an isomorphism $\varphi : H \rightarrow G_0$ such that, for all $h \in H$ and for all complex irreducible characters $\chi$ of $G$, $\chi(h) = \chi(\varphi(h))$. Then $G_0 \cong H$.

Proof. This has been proved when $H$ is a cyclic subgroup in [4, 5]. The same argument gives a proof in general. See also [7, Lemma 4.6].

We can now prove the main result of this note.

Proof of the theorem. Let $G = \langle a \rangle \times X$, where $o(a) = n$, $|X| = m$, and $(n, m) = 1$; also let $H$ be a finite subgroup of $U_1ZG$. In order to prove the theorem we will construct an isomorphism $\varphi$ of $H$ onto a subgroup of $G$ satisfying the criterion of Lemma 4; that is, $\varphi$ will be such that, for all irreducible characters $\chi$ of $G$, $\chi(h) = \chi(\varphi(h))$ for all $h \in H$.

By Whitcomb's argument given on [6, p. 103] it follows that, for all $h \in H$, there exists an element $g_h$ of $G$ such that $h \equiv g_h$ (mod $\Delta G \Delta (a)$). Since in the metabelian group case being considered it has been proved by Cliff, Sehgal, and Weiss in [1] that $U(1 + \Delta(G, \langle a \rangle))$ is torsion-free, it follows that the torsion subgroup $H$ of $U_1ZG$ is isomorphic to a subgroup $G_0$ of $G$. Let $\alpha : H \rightarrow G_0$ be the above isomorphism defined by $\alpha(h) = g_h$. By Lemma 3, $G_0 = \langle b \rangle \times X_0$ where $b \in \langle a \rangle$ and $X_0$ is a group isomorphic to a subgroup of $X$. Taking preimages, we can write $H = \langle u \rangle \times K$, where $a(u) = b$ and $a(K) = X_0$.

Since $o(u)$ divides $o(a)$, by [2, Theorem 1.1], we have that $u \sim g$ for some $g \in \langle a \rangle$. Also, since $\alpha$ is an isomorphism and $u \sim g$, we have $o(g) = o(u) = o(b)$. Hence $g = b_i$ for some $i$ and $u \sim b_i$.

Now let $k \in K$. We have $\alpha(k) = a^{j_1} x$ for some $j$ and some $x \in X$. Note that if $k_1 \neq k_2$ and $\alpha(k_1) = a^{j_1} x$, $\alpha(k_2) = a^{j_2} x$, then $x_1 \neq x_2$ (since otherwise $\alpha(k_1 k_2^{-1}) \in \langle a \rangle$, contradicting the relative primeness of $|K|$ and $o(a)$).

We shall prove by induction on the number of different primes dividing $o(a)$ that $k \sim x$. Since $k \equiv a^j x$ (mod $\Delta G \Delta (a)$), it follows that $k = (1 + \delta)x$ for some $\delta \in \Delta G \Delta (a)$. Hence, if $\langle a \rangle$ is a $p$-group, since $(o(k), o(a)) = 1$, we obtain, by Lemma 1, that $k \sim x$, and we are done in this case.

Now let in general $o(a) = p_1^{n_1} \cdots p_r^{n_r}$, and write $\langle a \rangle = \langle a_1 \rangle \times \langle a_2 \rangle$, where $o(a_1) = p_1^{n_1}$ and $o(a_2) = p_2^{n_2} \cdots p_r^{n_r}$. Then $G = \langle a_1 \rangle \times G_2$, where $G_2 = \langle a_2, X \rangle$, and we have that $U_1ZG = U(1 + \Delta(G, \langle a_1 \rangle)) \times U_1ZG_2$. If we write $k = (1 + \delta_1)\gamma$, where $(1 + \delta_1) \in U(1 + \Delta(G, \langle a_1 \rangle))$ and $\gamma \in U_1ZG_2$, then, by Lemma 1, it follows that $k \sim \gamma$. On the other hand, we can write $\gamma = (1 + \delta_2)\gamma$, where $(1 + \delta_2) \in U(1 + \Delta(G_2, \langle a_2 \rangle))$ and $\gamma \in U_1ZX = X$. But then by the inductive hypothesis $\gamma \sim \gamma$, and this forces $k \sim \gamma$. We now claim $x = \gamma$. In fact, since $(1 + \delta_1) \equiv a_1^t$ (mod $\Delta G \Delta (a_1)$) for some $t$ and $\gamma = (1 + \delta_2)\gamma \equiv a_1^t \gamma$ (mod $\Delta G_2 \Delta (a_2)$) for some $s$, we have $k = (1 + \delta_1)(1 + \delta_2)\gamma \equiv a_1^t a_2^s \gamma$ (mod $\Delta G \Delta (a)$). In particular, $a_1^t a_2^s \gamma \equiv a^t x$ (mod $\Delta G \Delta (a)$). By [1] it follows that $a_1^t a_2^s \gamma = a^t x$, and so $x = \gamma$, as claimed.
We now define a map \( \varphi : H \to G \) by setting \( \varphi(u) = b^i \) and \( \varphi(k) = x \), where \( k \in K \) and \( \alpha(k) = a^j x \) for some \( j \). It is easy to see that \( \varphi \) is a homomorphism, and our earlier argument tells us that \( \varphi \) is one-to-one. The conjugacy results obtained above allow us to conclude that \( \chi(u) = \chi(\varphi(u)) \) and \( \chi(k) = \chi(\varphi(k)) \) for all \( k \in K \) and all irreducible characters \( \chi \) of \( G \).

We want to prove that, for every \( h \in H \) and for every irreducible character \( \chi \) of \( G \), \( \chi(h) = \chi(\varphi(h)) \). With this, the proof of the theorem will be completed according to Lemma 4. To this effect we use induction on the number of different primes dividing \( \alpha(a) \) but not dividing \( \alpha(h) \).

Let \( h = u'k \in H \) for some \( k \in K \). If \( \alpha(h) \) is divisible by all primes dividing \( \alpha(a) \), then, since \( \alpha(b^{it}x) = \alpha(\varphi(h)) = \alpha(h) \), by Lemma 2, \( x \) is central in \( G \). Hence, since \( k \sim x \), \( k = x \). Thus \( h = u'k \) and \( \varphi(h) = b^{it}x \) are conjugate, and this says that \( \chi(h) = \chi(\varphi(h)) \) for all irreducible characters \( \chi \) of \( G \). Therefore, we may assume that at least one of the primes dividing \( \alpha(a) \), say \( p \), does not divide \( \alpha(h) \).

As above write \( \langle a \rangle = \langle a_1 \rangle \times \langle a_2 \rangle \), where \( \langle a_1 \rangle \) is a \( p \)-group and \( \langle a_2 \rangle \) is a \( p' \)-group. Then \( G = \langle a_1 \rangle \rtimes G_2 \), where \( G_2 = \langle a_2, x \rangle \), and we have \( U_1 \mathbb{Z}G = U(1 + \Delta(G, \langle a_1 \rangle)) \rtimes U_1 \mathbb{Z}G_2 \). Moreover, \( h = (1 + \delta)\gamma \), where \( (1 + \delta) \in U(1 + \Delta(G, \langle a_1 \rangle)) \) and \( \gamma \in U_1 \mathbb{Z}G_2 \). By Lemma 1, it follows that \( h \sim \gamma \), so \( \chi(h) = \chi(\gamma) \) for all irreducible characters \( \chi \) of \( G \).

Let us denote by \( \overline{\varphi} \) the homomorphism induced by \( \varphi \) when we factorize by \( \langle a_1 \rangle \), i.e.,

\[
\overline{\varphi} : H = \langle \overline{u} \rangle \rtimes \overline{K} \to \overline{G} = G_2.
\]

Our map behaves well with respect to factoring by \( \langle a_1 \rangle \); namely \( \overline{\varphi}(\overline{u}) = \overline{b}^i \)
and \( \overline{\varphi}(\overline{k}) = \overline{x} \).

Since \( h = (1 + \delta)\gamma \), where \( (1 + \delta) \in U(1 + \Delta(G, \langle a_1 \rangle)) \) and \( \gamma \in U_1 \mathbb{Z}G_2 \), by factoring by \( \langle a_1 \rangle \) we get \( \overline{h} = \gamma \), so \( \overline{\varphi}(\overline{h}) = \overline{\varphi}(\overline{\gamma}) = \overline{b}^{it} \overline{x} \).

By the inductive hypothesis, \( \psi(\overline{\varphi}(\overline{h})) = \psi(\overline{h}) = \psi(\gamma) = \psi(\overline{b}^{it} \overline{x}) \) for every irreducible character \( \psi \) of \( G_2 = A_2 \rtimes X \). But if \( \chi \) is an irreducible character of \( G \) and \( \chi_{G_2} \) is the induced character on \( G_2 \), then \( \chi_{G_2} \) is a linear combination of irreducible \( G_2 \)-characters, and by the previous argument it follows that \( \chi(\gamma) = \chi_{G_2}(\gamma) = \chi_{G_2}(\overline{b}^{it} \overline{x}) = \chi(\overline{b}^{it} \overline{x}) \). Finally, by writing \( b^{it}x = a_1^{l} \overline{b}^{it} \overline{x} \), for some \( l \), since \( p \) does not divide \( \alpha(h) = \alpha(b^{it}x) \), it follows, by Lemma 1, that \( b^{it}x \sim \overline{b}^{it} \overline{x} \). Thus \( \chi(h) = \chi(\gamma) = \chi(\overline{b}^{it} \overline{x}) = \chi(b^{it}x) \) for every irreducible character \( \chi \) of \( G \). This completes the proof of the theorem.

**References**


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**Dipartimento di Matematica ed Applicazioni, Università di Palermo, Via Archirafi 34, 90123 Palermo, Italy**