STRONG MOMENT PROBLEMS
FOR RAPIDLY DECREASING SMOOTH FUNCTIONS

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Abstract. It is shown that the existence of rapidly decreasing smooth solutions of various moment problems follows from the theorem of Ritt on the existence of analytic functions with a prescribed asymptotic power series at the vertex of a given sector.

1. Introduction

The problem of moments, as well as its many generalizations, have been the object of research for over a century. In its basic form it consists of finding a positive function \( \phi(x) \), defined on a given interval \( I \), that satisfies

\[
\int_I x^n \phi(x) \, dx = \mu_n, \quad n \in \mathbb{N},
\]

where \( \{\mu_n\}_{n \in \mathbb{N}} \) is a given sequence of real or complex numbers. We refer to [1, 17] for an account of the classical aspects of this problem.

The moments play a rather important role in several areas of current interest, including the theory of asymptotic expansions of generalized functions [7, 8, 10], the theory of orthogonal polynomials [12, 14], and the theory of distributional solutions of differential and functional equations [13, 19].

Recently, Durán [6] proved that when \( I = [0, \infty) \), for an arbitrary sequence \( \{\mu_n\}_{n \in \mathbb{N}} \), problem (1.1) admits solutions of the class \( \mathcal{S}(0, \infty) = \{ \phi \in \mathcal{S} : \phi(x) = 0, x \leq 0 \} \). Here \( \mathcal{S} = \mathcal{S}(\mathbb{R}) \) is the Schwartz space of rapidly decreasing smooth functions [16]; namely,

\[
\mathcal{S}(\mathbb{R}) = \left\{ \phi \in C^\infty(\mathbb{R}) : \lim_{|x| \to \infty} x^n \phi^{(m)}(x) = 0, \, n, \, m \in \mathbb{N} \right\}.
\]

The arbitrariness of the sequence \( \{\mu_n\}_{n \in \mathbb{N}} \) had already been obtained by Boas [4], who worked in the class of functions of bounded variation. This contrasts with the situation when \( I \) is a compact interval, since growth conditions on certain linear combinations of the \( \mu_n \)'s are required for existence, even in the case when \( \phi \) is a distribution [9].
In the present article we show that this result as well as the existence of solutions of more general problems such as the strong moment problem,

\begin{align}
\int_{-\infty}^{\infty} x^n \phi(x) \, dx = \mu_n, \quad n \in \mathbb{Z},
\end{align}

can be obtained from the following theorem of Ritt [18, p. 41]:

**Theorem.** Let \( \{b_n\}_{n \in \mathbb{N}} \) be a sequence of complex numbers. Then there exists a bounded analytic function \( F(z) \) defined in the sector \( S_\alpha(\alpha, \beta) = \{re^{i\theta} : 0 < r < a, \alpha < \theta < \beta \} \) with the asymptotic power series

\begin{align}
F(z) \sim \sum_{n=0}^{\infty} b_n z^n, \quad z \to 0, \quad z \in S_\alpha(\alpha, \beta). \quad \square
\end{align}

Actually, the function \( F(z) \) can be chosen analytic and bounded in the whole sector \( S(\alpha, \beta) = S_\infty(\alpha, \beta) = \{re^{i\theta} : r > 0, \alpha < \theta < \beta \} \). This can be obtained from the result for \( S_\alpha(\alpha, \beta), \quad a < \infty \), as follows. Write \( F(z) = f(h(z)) \), where \( h : S(\alpha, \beta) \to S_\alpha(\alpha, \beta) \) is a conformal map with \( h(0) = 0 \) and \( h(z) \sim z + c_2 z^2 + c_3 z^3 + \cdots \), as \( z \to 0 \), and where \( f(\omega) \) is chosen as in the theorem with asymptotic development \( f(\omega) \sim d_0 + d_1 \omega + d_2 \omega^2 + \cdots \), as \( \omega \to 0 \), with the \( d_i \)'s chosen in such a way that, formally,

\begin{align}
\sum_{j=0}^{\infty} d_j \left( z + \sum_{n=2}^{\infty} c_n z^n \right)^j = \sum_{n=0}^{\infty} b_n z^n.
\end{align}

### 2. Solution of the Moment Problem

In this section we use the theorem of Ritt to give a proof of the existence of solutions in the class \( \mathcal{E}(0, \infty) \) of the problem

\begin{align}
\int_{0}^{\infty} x^n \phi(x) \, dx = \mu_n, \quad n \in \mathbb{N},
\end{align}

where \( \{\mu_n\}_{n \in \mathbb{N}} \) is an arbitrary sequence. This result was obtained by Durán [6], who gave a different proof.

We first recall [2] that a function \( \psi \in \mathcal{E}(\mathbb{R}) \) is the Fourier transform

\begin{align}
\psi(u) = \hat{\phi}(u) = \int_{0}^{\infty} e^{ixu} \phi(x) \, dx
\end{align}

of a function \( \phi \) of the class \( \mathcal{E}(0, \infty) \) if and only if it can be extended to a bounded continuous function \( \Psi(z) \) in the upper half plane \( \text{Im} \, z \geq 0 \), analytic in \( \text{Im} \, z > 0 \) and vanishing as \( z \to \infty \).

Since

\begin{align}
\frac{d^n \hat{\phi}(0)}{du^n} = i^n \int_{0}^{\infty} x^n \phi(x) \, dx,
\end{align}

problem (2.1) is equivalent to that of finding \( \psi = \hat{\phi} \) in the class \( \mathcal{F}(\mathcal{E}(0, \infty)) \) that satisfies

\begin{align}
\psi^{(n)}(0) = i^n \mu_n, \quad n \in \mathbb{N}.
\end{align}
We construct \( \psi \) as follows. Let
\[
G(z) = e^{(1-i)(z+i)^{1/2}},
\]
where the branch of the square root is chosen so as to assure that \( G(z) \to 0 \) as \( z \to \infty \) in the upper half plane \( \text{Im} \ z \geq 0 \). Let the sequence \( a_0, a_1, a_2, \ldots \) be defined as
\[
\frac{1}{G(z)} = a_0 + a_1 z + a_2 z^2 + \cdots, \quad |z| < 1.
\]
Let \( \delta \in (0, \pi/2) \), and let \( F(z) \) be a bounded analytic function in the sector \( S: -\delta < \text{Arg} \ z < \pi + \delta \) with the asymptotic power series
\[
F(z) \sim b_0 + b_1 z + b_2 z^2 + \cdots, \quad z \to 0, \ z \in S,
\]
where
\[
b_n = \sum_{k=0}^{n} \frac{i^k}{k!} \mu_k a_{n-k}.
\]
Let \( \Psi(z) = F(z)G(z) \). Then \( \Psi \) is analytic in the sector \( S \), and \( \Psi(z) = o(z^{-n}) \) for each \( n \in \mathbb{N} \) as \( z \to \infty \) within \( S \). Also,
\[
\Psi(z) \sim \sum_{n=0}^{\infty} \frac{i^n \mu_n z^n}{n!}, \quad z \to 0, \ z \in S.
\]
It follows that, if \( \psi \) is the restriction of \( \Psi \) to the real axis, then \( \psi \in \mathcal{S}(\mathcal{S}(0, \infty)) \) and \( \psi \) satisfies (2.4), as required.

3. The strong moment problem

In this section we consider the following strong moment problem: Find \( \phi \in \mathcal{S}(\mathbb{R}) \) that satisfies \( \phi^{(n)}(0) = 0, \ n \in \mathbb{N} \), and
\[
\int_{-\infty}^{\infty} x^n \phi(x) \, dx = \mu_n, \quad n \in \mathbb{Z},
\]
where \( \{\mu_n\}_{n \in \mathbb{Z}} \) is an arbitrary sequence. We refer to [3, 11, 15] for the study of this problem in the class of the positive Radon measures.

Let \( \psi(u) = \phi(u) \). Then \( \psi \in \mathcal{S}(\mathbb{R}) \), and (3.1) is equivalent to the two sets of conditions
\[
\begin{align*}
(3.2a) \quad \psi^{(n)}(0) &= i^n \mu_n, \quad n = 0, 1, 2, \ldots, \\
(3.2b) \quad \int_{0}^{\infty} \mu^n \psi(\mu) \, d\mu &= n! i^{n+1} \mu_{-n-1}, \quad n = 0, 1, 2, \ldots.
\end{align*}
\]
Using the construction of §2, we can find \( \psi_1 \in \mathcal{S}(\mathbb{R}) \) such that \( \psi_1^{(n)}(0) = i^n \mu_n, \ n = 0, 1, 2, \ldots \). We can also find \( \psi_2 \in \mathcal{S}(0, \infty) \) such that
\[
\int_{0}^{\infty} \mu^n \psi_2(\mu) \, d\mu = n! i^{n+1} \mu_{-n-1} - \int_{0}^{\infty} \mu^n \psi_1(\mu) \, d\mu.
\]
Then \( \psi = \psi_1 + \psi_2 \) belongs to \( \mathcal{S}(\mathbb{R}) \) and satisfies (3.2a,b). However, if \( \hat{\phi} = \psi \), the condition \( \phi^{(n)}(0) = 0, \ n \in \mathbb{N} \), might not be satisfied. Following a suggestion of Professor C. Berg, we solve this by choosing \( \psi_3 \in \mathcal{S}(-\infty, 0) \) such that
\[
\int_{-\infty}^{0} y^n \psi_3(y) \, dy = -n! i^{n+1} \mu_{-n-1} - \int_{-\infty}^{0} y^n \psi_1(y) \, dy,
\]
for \( n \in \mathbb{N} \), and define \( \psi = \psi_1 + \psi_2 + \psi_3 \). This function \( \psi \) satisfies (3.2a, b) but also

\[
\int_{-\infty}^{\infty} y^n \psi(y) \, dy = 0, \quad n \in \mathbb{N},
\]

and thus if \( \hat{\phi} = \psi \), then \( \phi^{(n)}(0) = 0 \), \( n \in \mathbb{N} \).

Actually, the strong moment problem can be solved in \( \mathcal{S}(0, \infty) \); that is, there are \( \sigma \in \mathcal{S}(0, \infty) \) with

\[
\int_{-\infty}^{\infty} x^n \sigma(x) \, dx = \mu_n, \quad n \in \mathbb{Z}.
\]

To see it, let \( \{\mu_n\}_{n \in \mathbb{Z}} \) be any sequence, and let \( \phi \in \mathcal{S}(\mathbb{R}) \) with \( \phi^{(n)}(0) = 0 \), \( n \in \mathbb{N} \), be a solution of the strong problem

\[
\int_{-\infty}^{\infty} x^n \phi(x) \, dx = \begin{cases} 
\mu_{n/2}, & \text{if } n \text{ even}, \\
0, & \text{if } n \text{ odd}.
\end{cases}
\]

Let \( \rho \in \mathcal{S}(0, \infty) \) be given by \( \rho(x) = \phi(x) + \phi(-x) \), \( x \geq 0 \). Then

\[
\int_{0}^{\infty} u^{2n} \rho(u) \, du = \mu_n, \quad n \in \mathbb{Z}.
\]

Set

\[
\sigma(x) = \frac{\rho(x^{1/2})}{2x^{1/2}}, \quad x \geq 0.
\]

Then \( \sigma \in \mathcal{S}(0, \infty) \), and it satisfies (3.6).

Summarizing, we have shown the following result:

**Theorem.** Let \( \{\mu_n\}_{n \in \mathbb{Z}} \) be an arbitrary sequence. Then there exist functions \( \phi \in \mathcal{S}(0, \infty) \) that satisfy

\[
\int_{0}^{\infty} x^n \phi(x) \, dx = \mu_n
\]

for each \( n \in \mathbb{Z} \).

### 4. Further remarks

Our results imply the existence of smooth functions \( \phi_k \in \mathcal{S}(0, \infty) \), \( k = 0, 1, 2, \ldots \), that satisfy the moment problem

\[
\int_{0}^{\infty} x^n \phi_k(x) \, dx = \delta_{nk},
\]

where \( \delta_{nk} = 1 \) if \( n = k \), \( \delta_{nk} = 0 \) if \( n \neq k \) is the Kronecker delta. The functions \( \phi_k(x) \) can be used as “smooth versions” of the Dirac delta functions \((-1)^k \delta^{(k)}(x)/k!\), which also satisfy

\[
\left\langle \frac{(-1)^k \delta^{(k)}(x)}{k!}, x^n \right\rangle = \delta_{nk}.
\]

In particular, if \( \psi \in \mathcal{S}(\mathbb{R}) \) has moments \( \mu_n = \langle \psi(x), x^n \rangle \), \( n = 0, 1, 2, \ldots \), then we could expect some relationship between \( \psi(x) \) and the series...
\[ \sum_{n=0}^{\infty} \mu_n \phi_n(x) \]. Although the series is not necessarily equal to \( \psi \) (it could even be divergent), we do have the asymptotic relation

\[ \psi(\lambda x) \sim \mu_0 \phi_0(\lambda x) + \mu_1 \phi_1(\lambda x) + \mu_2 \phi_2(\lambda x) + \cdots, \quad \text{as } \lambda \to \infty \]

in the space \( \mathcal{S}'(\mathbb{R}) \). The interpretation of (4.3) is in the distributional sense; namely, that for each \( \rho \in \mathcal{S}(\mathbb{R}) \) we have

\[ \int_{-\infty}^{\infty} \psi(\lambda x) \rho(x) \, dx = \sum_{j=0}^{n} \mu_j \int_{-\infty}^{\infty} \phi_j(\lambda x) \rho(x) \, dx + O(\lambda^{-n-2}), \quad \text{as } \lambda \to \infty. \]

Relation (4.3) can be interpreted as saying that \( \{\phi_k\}_{k \in \mathbb{N}} \) is an “asymptotic basis” of the space \( \mathcal{S}'(\mathbb{R}) \). Actually, (4.3) is a smooth version of the moment asymptotic expansion \([10]\)

\[ f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n! \lambda^{n+1}}, \quad \text{as } \lambda \to \infty, \]

valid if \( f \) is a generalized function of distributional rapid decay at infinity, where the \( \mu_n = (f(x), x^n) \) are the moments.

We would like to remark that our results imply that the class \( \mathcal{M} \) of entire functions of the form

\[ \Phi(z) = \int_{0}^{\infty} x^z \phi(x) \, dx, \]

where \( \phi \in \mathcal{S}(0, \infty) \), satisfies the following interpolation property: for each sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \) there are \( \Phi \in \mathcal{M} \) that satisfy

\[ \Phi(n) = \mu_n, \quad n \in \mathbb{Z}. \]

The class \( \mathcal{M} \) is larger than the class of entire functions of exponential type \([5]\). Indeed, if \( \Phi \in \mathcal{M} \) then there exists \( \sigma \in C^\infty(\mathbb{R}) \) such that \( \sigma(t) = o(e^{-n|t|}) \) as \( |t| \to \infty \) for each \( n \in \mathbb{N} \), with

\[ \Phi(z) = \int_{-\infty}^{\infty} e^{zt} \sigma(t) \, dt. \]

For functions of exponential type, the \( \sigma(t) \) has compact support. Actually, there are sequences \( \{\mu_n\}_{n \in \mathbb{Z}} \) for which (4.7) does not have solutions of exponential type.

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REFERENCES


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