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ANALYTIC CONTINUATION OF RIEMANN'S ZETA FUNCTION AND VALUES AT NEGATIVE INTEGERS VIA EULER'S TRANSFORMATION OF SERIES

JONATHAN SONDOW

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ABSTRACT. We prove that a series derived using Euler's transformation provides the analytic continuation of $\zeta(s)$ for all complex $s \neq 1$. At negative integers the series becomes a finite sum whose value is given by an explicit formula for Bernoulli numbers.

1. INTRODUCTION

Euler computed the values of the zeta function at the negative integers using both Abel summation (75 years before Abel) and the Euler-Maclaurin sum formula. (Comparison of these values with those he found at the positive even integers led him to conjecture the functional equation 100 years before Riemann!) Euler also used a third method, his transformation of series or (E) summation (see §2), to calculate $\zeta(-n)$, but only for $n = 0, 1, 2$, and 4. (See [1; 4, §1.5; 5, volume 14, pp. 442–443, 594–595; volume 15, pp. 70–90; 7; 9, §§1.3, 1.6, 2.2, 2.3; 10; 14, Chapter III, §§XVII–XX].)

We observe in §4 that this last method in fact yields

$$\zeta(-n) = (-1)^n B_{n+1}/(n+1) \quad \text{for all } n \geq 0,$$

but we require an explicit formula for Bernoulli numbers that was discovered a century after Euler. In §3 we justify the method by proving that a series used in §4 gives the analytic continuation of $\zeta(s)$ for all $s \neq 1$. Similar results for approximations to Euler's transformation are obtained in §5, as well as an evaluation of $\zeta'(0)/\zeta(0) = \log 2\pi$.

In a paper in preparation, the author will apply the method to other zeta functions and to Dirichlet L-series.

2. EULER'S TRANSFORMATION OF SERIES

Any convergent series of complex numbers, written with alternating signs as

$$A = a_1 - a_2 + a_3 - \cdots,$$

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can also be written in the form

$$A = \tfrac{1}{2}a_1 + \tfrac{1}{2}[(a_1 - a_2) - (a_2 - a_3) + \cdots].$$

Repeating the process on the series in brackets, we have

$$A = \tfrac{1}{2}a_1 + \tfrac{1}{4}(a_1 - a_2) + \tfrac{1}{4}[(a_1 - 2a_2 + a_3) - (a_2 - 2a_3 + a_4) + \cdots]$$

and in general

$$\sum_1^\infty (-1)^{n-1} a_n = \sum_0^{k-1} \frac{\Delta^j a_1}{2^{j+1}} + \sum_{n=1}^\infty (-1)^{n-1} \frac{\Delta^k a_n}{2^k}, \tag{1}$$

where $\Delta^0 a_n = a_n$ and

$$\Delta^k a_n = \Delta^{k-1} a_n - \Delta^{k-1} a_{n+1} = \sum_{m=0}^{k} (-1)^m \binom{k}{m} a_{n+m}$$

for $k \geq 1$. It is proved in [11, §33B] that the sum of the last series in (1) approaches 0 as $k \to \infty$, so that

$$\sum_1^\infty (-1)^{n-1} a_n = \sum_0^\infty \frac{\Delta^j a_1}{2^{j+1}}, \tag{2}$$

which is *Euler's transformation of series.* (See [5, volume 10, pp. 222–227; 9, §4.6; 11, §§35B, 59, 63].)

3. Analytic continuation of $\zeta(s)$

Instead of working directly with $\zeta(s)$, which for $\sigma = \mathrm{Re}(s) > 1$ is given by $\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \cdots$, let us consider the alternating series

$$\zeta(s) - 2 \cdot 2^{-s}\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - \cdots, \tag{3}$$

which converges for $\sigma > 0$ (see §5). Applying the Euler transformation, we have, for $\sigma > 1$,

$$\begin{aligned} (1 - 2^{1-s})\zeta(s) &= \sum_0^\infty \frac{\Delta^j 1^{-s}}{2^{j+1}} \\ &= \sum_0^\infty \frac{1 - \binom{j}{1}2^{-s} + \binom{j}{2}3^{-s} - \cdots + (-1)^j \binom{j}{j}(j+1)^{-s}}{2^{j+1}}. \end{aligned} \tag{4}$$

Theorem. *The analytic continuation of $\zeta(s)$ for all complex $s \neq 1$ is given by the product*

$$\zeta(s) = (1 - 2^{1-s})^{-1} \sum_0^\infty \frac{\Delta^j 1^{-s}}{2^{j+1}} \tag{5}$$

in which the series converges absolutely and uniformly on compact sets to an entire function.

Proof. Fix $k \geq 0$. Evidently

$$\Delta^k n^{-s} = (s)_k \int_0^1 \cdots \int_0^1 (n + x_1 + \cdots + x_k)^{-s-k}\, dx_1 \cdots dx_k$$

for $k = 1, 2, \dots$, where $(s)_k$ denotes the product $s(s+1)\cdots(s+k-1)$. Hence,

$$|\Delta^k n^{-s}| \le |(s)_k|/n^{\sigma+k} \quad \text{whenever } \sigma + k \ge 0, \tag{6}$$

$k = 0, 1, 2, \dots$, where $(s)_0 = 1$. Now let S be a compact set in the half plane $\sigma > 1-k$, and let M_n denote the maximum of $|(s)_k|/n^{\sigma+k}$ on S. Then (6) implies that $\sum M_n$ dominates the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\Delta^k n^{-s} \tag{7}$$

on S. It follows, using the triangle inequality, that the Euler transform of $\sum M_n$ dominates the Euler transform of (7), which, since $\Delta^j\Delta^k = \Delta^{j+k}$, is

$$\sum_{j=0}^{\infty}\frac{\Delta^j\Delta^k 1^{-s}}{2^{j+1}} = \sum_{j=k}^{\infty}\frac{\Delta^j 1^{-s}}{2^{j+1-k}}.$$

Multiplying this by $1/2^k$ and adding $\sum_0^{k-1}\Delta^j 1^{-s}/2^{j+1}$ produces the series in (4), which, since k is arbitrary, therefore converges absolutely and uniformly on compact sets to an entire function. Since the series in (3) has zeros at the (simple) poles of $(1-2^{1-s})^{-1}$ except at $s = 1$ (for a direct proof see [12]), the theorem follows.

4. Evaluation of $\zeta(-m)$

Let m be a positive integer or 0. Note that $(-m)_j = 0$ and, hence, $\Delta^j 1^m = 0$ for $j > m$. Thus when $s = -m$ the series in (5) becomes a finite sum. Its value is given by a formula for Bernoulli numbers that Carlitz [2] attributes to Worpitzky [15] (see also [3]), namely, the second equality in the following.

Corollary. *For* $m = 0, 1, 2, \dots$,

$$\zeta(-m) = \frac{1}{1-2^{m+1}}\sum_0^m \frac{\Delta^j 1^m}{2^{j+1}} = (-1)^m\frac{B_{m+1}}{m+1}.$$

Alternatively, one can view $\zeta(-m) = (-1)^m B_{m+1}/(m+1)$ as known, which gives a proof of Worpitzky's formula (compare [6]).

5. Approximations to Euler's transformation

Note that (1), (3), and (6) imply (without using (2)) that *for* $k \ge 1$ *the product*

$$\zeta(s) = (1-2^{1-s})^{-1}\left(\sum_0^{k-1}\frac{\Delta^j 1^{-s}}{2^{j+1}} + \frac{1}{2^k}\sum_{n=1}^{\infty}(-1)^{n-1}\Delta^k n^{-s}\right) \tag{8}$$

provides the analytic continuation of $\zeta(s)$ *on the punctured half plane* $\sigma > 1-k$, $s \ne 1$ *where the infinite series converges absolutely and uniformly on compact sets to a holomorphic function. Moreover, except that the convergence will not be absolute in the strip* $-k < \sigma \le 1-k$, *this remains true for* $k \ge 0$ *and* $\sigma > -k$,

$s \neq 1$. (*Proof.* Grouping terms in pairs in the even partial sums of the second summation, we have

$$\sum_{n=1}^{2N}(-1)^{n-1}\Delta^k n^{-s} = \sum_{n=1}^{N}(\Delta^k(2n-1)^{-s} - \Delta^k(2n)^{-s})$$
$$= \sum_{n=1}^{N}\Delta^{k+1}(2n-1)^{-s}.$$

Then it follows from (6) that both even and odd partial sums converge as required.) Since we can use (8) with $k \geq m+1$ to evaluate $\zeta(-m)$, the approximations (1) to Euler's transformation yield everything it does except formula (5).

As an example, take $k = 1$ in (8):

$$\zeta(s) = (1 - 2^{1-s})^{-1}\left(\frac{1}{2} + \frac{1}{2}\sum_{1}^{\infty}(-1)^{n-1}(n^{-s} - (n+1)^{-s})\right)$$

for $\sigma > -1$, $s \neq 1$. (This formula appears in Hardy's proof of the functional equation [8; 13, §2.2] and gave the idea for the present note.) Thus $\zeta(0) = -1/2$ and, using Wallis's product for $\pi/2$,

$$\frac{\zeta'(0)}{\zeta(0)} = 2\log 2 + \log\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\cdots = \log 2\pi,$$

which figures in the Hadamard product representation of the zeta function.

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Department of Mathematics, Yeshiva University, New York, New York 10033
Current address: 209 West 97th St., New York, New York, 10025