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ON THE C^*-ALGEBRA GENERATED BY THE LEFT REGULAR REPRESENTATION OF A LOCALLY COMPACT GROUP

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ABSTRACT. Let λ denote the left regular representation of a locally compact group G on $L^2(G)$ and $C^*(\lambda(G))$ the C^*-algebra generated by $\lambda(G)$. We show that the amenability of G and the amenability of G considered as a discrete group both may be characterized in terms of $C^*(\lambda(G))$.

1. INTRODUCTION

We first fix some notation. Throughout this note we let G denote a locally compact (Hausdorff topological) group equipped with a fixed left Haar measure μ and G_d denote the group G considered as a discrete group. As usual, $L^1(G)$, $L^2(G)$, and $L^\infty(G)$ are defined with respect to μ. The left regular representation of G on $L^2(G)$, defined by

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h), \qquad \xi \in L^2(G), \ g, h \in G,$$

is well known to be a (strongly) continuous unitary representation of G. We shall denote by λ_d the left regular representation of G_d on $l^2(G_d)$. All undefined terminology in this paper is explained in at least one of the references [5, 8, 11, 15, 17, 18].

Much attention has been devoted to the study of the following operator algebras associated with G: the full group C^*-algebra $C^*(G)$, the reduced group C^*-algebra $C_r^*(G)$, and the group von Neumann algebra $vN(G)$. We recall that $C^*(G)$ is defined as the enveloping C^*-algebra of $L^1(G)$ considered as an involutive Banach algebra with an approximate identity. If $\mathscr{B}(L^2(G))$ denotes the bounded linear operators on $L^2(G)$, then $C_r^*(G)$ is the C^*-subalgebra of $\mathscr{B}(L^2(G))$ generated by the convolution operators T_f, $f \in L^1(G)$, where $T_f(\xi) = f * \xi$, $\xi \in L^2(G)$. At last, $vN(G)$ is the von Neumann subalgebra of $\mathscr{B}(L^2(G))$ generated by $\lambda(G) = \{\lambda(g), \ g \in G\}$ or, equivalently, $vN(G) = \lambda(G)'' = C_r^*(G)''$, where $''$ denotes the double commutant (in $\mathscr{B}(L^2(G))$).

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The purpose of this note is to draw attention to $C^*(\lambda(G))$, the C^*-subalgebra of $\mathscr{B}(L^2(G))$ generated by $\lambda(G)$. Of course, when G is discrete, we have $C^*(\lambda(G)) = C_r^*(G)$, and we will therefore mainly be interested in the nondiscrete case. In this case, it is known that $C_r^*(G)$ and $C^*(G)$ are nonunital [14, Corollaries 1 and 2], while $C^*(\lambda(G))$ is always unital.

The first paper we are aware of which explicitly deals with $C^*(\lambda(G))$ in the nondiscrete case is [12], where Kodaira and Kakutani essentially show that, when G is abelian, $C^*(\lambda(G))$ is $*$-isomorphic to $\mathscr{C}(\widehat{G}_d)$, the continuous complex functions on the dual group of G_d. This result is nicely exposed by Arveson in [1], where he generalizes it to other C^*-algebras generated by abelian unitary groups. Further, when G is abelian, it is well known that $C^*(G) \simeq C_r^*(G) \simeq \mathscr{C}_0(\widehat{G})$, the continuous complex functions on the dual group of G which vanish at infinity. Thus, $C^*(\lambda(G))$ on one hand and $C^*(G) \simeq C_r^*(G)$ on the other hand contain rather different information in the abelian case. Also, still in this case, we have $C_r^*(G_d) \simeq \mathscr{C}(\widehat{G}_d)$; hence, $C^*(\lambda(G)) \simeq C_r^*(G_d)$. Further, Zeller-Meier showed in [21] that the same conclusion is true whenever G_d is amenable (see also [3, 4, 9]). One may therefore wonder whether all the topological flavour of G always disappears in $C^*(\lambda(G))$.

We shall show that this suggestion is not generally true. Our approach relies heavily on the now well-developed theory of amenability [17, 18]. We recall that G is called amenable whenever there exists a left invariant mean on $L^\infty(G)$, i.e., a state on $L^\infty(G)$ which is invariant under left translations. A deep C^*-algebraic characterization of the amenability of G is that $C^*(G)$ and $C_r^*(G)$ are canonically $*$-isomorphic ([17, Theorem 4.21] or [18, Theorem 8.9]). Another characterization via $C^*(\lambda(G))$ is possible: our first result (Theorem 1) is that G is amenable if and only if there exists a nonzero multiplicative linear functional on $C^*(\lambda(G))$. We notice that the "only if" part is known in the discrete case [6, Theorem 2; 16, Proof of Proposition 1.6], and that, as pointed out to us by Alain Valette, a stronger result than Theorem 1 may be obtained (see the comments following Theorem 1). Our result provides a natural C^*-explanation to the fact that an abelian group G is amenable: $C^*(\lambda(G))$ is then an abelian C^*-algebra and therefore possesses a nonzero multiplicative linear functional by Gelfand's theory. Of course, this is not the most efficient way to prove this fact, which is an easy consequence of the Markov-Kakutani fixed point theorem (cf. [17, Proposition 0.15]).

With the help of Theorem 1, we can complement the result of Zeller-Meier in [21] and conclude that G_d is amenable if and only if G is amenable and $C^*(\lambda(G)) \simeq C_r^*(G_d)$ (Theorem 3). Hence, if G is an amenable group such that G_d is not amenable (e.g., $G = \mathrm{SO}(3)$), then $C^*(\lambda(G))$ is not $*$-isomorphic to $C_r^*(G_d)$.

At last, we characterize the nuclearity of $C^*(\lambda(G))$. We recall that a C^*-algebra is called nuclear if there is a unique way of forming its tensor product with any other C^*-algebra. For some equivalent definitions, the reader may consult [13, 17, 19], where further references are given. As a sample of the work of many hands, we quote the following from [17, 1.31 and 2.35]:

> G is amenable if and only if G is inner amenable and $C_r^*(G)$ is nuclear, if and only if G is inner amenable and $vN(G)$ is injective.

Inner amenability of G means here that there exists a state on $L^\infty(G)$ invariant under the action on $L^\infty(G)$ by inner automorphisms of G, while $vN(G)$ is injective whenever there exists a norm one projection from $\mathcal{B}(L^2(G))$ onto $vN(G)$. We also recall that there exist nonamenable groups G such that $C_r^*(G)$ is nuclear and $vN(G)$ is injective. Now, since any discrete group is inner amenable in the above sense, we have that G_d is amenable if and only if $C_r^*(G_d)$ is nuclear, a result proved by Lance in [13, Theorem 4.2]. We shall use this to conclude that G_d is amenable if and only if $C^*(\lambda(G))$ is nuclear (Theorem 3). Especially, we get that if G is amenable but G_d is not, then $C^*(\lambda(G))$ is nonnuclear while $C_r^*(G)$ is nuclear and $vN(G)$ is injective.

2. The results

We first give a quite direct proof of the following result.

Theorem 1. *G is amenable if and only if there exists a nonzero multiplicative linear functional on $C^*(\lambda(G))$.*

Proof. Suppose G is amenable. Then there exists a net $\{\xi_i\}$ in $\{\xi \in L^2(G) \mid \|\xi\|_2 = 1\}$ such that

$$\|\lambda(g)\xi_i - \xi_i\|_2 \to 0 \quad \text{for all } g \text{ in } G$$

(cf. [17, Theorem 4.4] or [18, Corollary 6.15]). For each i, define φ_i on $C^*(\lambda(G))$ by

$$\varphi_i(x) = \langle x\xi_i, \xi_i\rangle, \qquad x \in C^*(\lambda(G)).$$

Then $\{\varphi_i\}$ is a net in the state space of $C^*(\lambda(G))$ which (by Banach-Alaoglu's theorem) is weak*-compact. Hence we may pick a weak*-limit point of this net, say φ, which is a state on $C^*(\lambda(G))$. Now, since

$$|\varphi_i(\lambda(g)) - 1| = |\langle(\lambda(g)\xi_i - \xi_i), \xi_i\rangle| \leq \|\lambda(g)\xi_i - \xi_i\|_2 \to 0 \quad \text{for all } g \text{ in } G,$$

we clearly have $\varphi(\lambda(g)) = 1$ for all g in G. From the Cauchy-Schwartz inequality for states, one then easily gets that φ is multiplicative at each $\lambda(g)$, $g \in G$. As $\lambda(G)$ generates $C^*(\lambda(G))$ by definition, it follows by density that φ is a nonzero multiplicative linear functional on $C^*(\lambda(G))$.

Conversely, suppose φ is such a functional on $C^*(\lambda(G))$. Then, as φ preserves adjoints [15, Proposition 2.1.9], φ is a state on $C^*(\lambda(G))$ such that $|\varphi(\lambda(g))| = 1$ for all g in G. By the Hahn-Banach theorem for states [5, Proposition 2.3.24], we may extend φ to a state $\tilde{\varphi}$ on $\mathcal{B}(L^2(G))$ which satisfies

$$|\tilde{\varphi}(\lambda(g))| = 1 \quad \text{for all } g \text{ in } G.$$

Again from the Cauchy-Schwartz inequality for states, it follows that $\tilde{\varphi}$ is multiplicative at each $\lambda(g)$, $g \in G$.

We then have

$$\begin{aligned}\tilde{\varphi}(\lambda(g)x\lambda(g^{-1})) &= \tilde{\varphi}(\lambda(g))\tilde{\varphi}(x\lambda(g^{-1})) = \tilde{\varphi}(\lambda(g))\tilde{\varphi}(x)\tilde{\varphi}(\lambda(g^{-1})) \\ &= |\tilde{\varphi}(\lambda(g))|^2\tilde{\varphi}(x) = \tilde{\varphi}(x)\end{aligned}$$

for all g in G and x in $\mathcal{B}(L^2(G))$.

The amenability of G follows readily from this in a quite standard way (cf. [2, Theorem 2.2] or [6, Theorem 2]). If M_f denotes the multiplication operator on $L^2(G)$ by $f \in L^\infty(G)$, then one obtains a left invariant mean m

on $L^\infty(G)$ by defining $m(f) = \tilde{\varphi}(M_f)$, $f \in L^\infty(G)$, and by using that $M_{f_g} = \lambda(g)M_f\lambda(g^{-1})$ for all f in $L^\infty(G)$ and g in G, where $f_g(h) = f(g^{-1}h)$, $h \in G$. □

When U is a continuous unitary representation of G on a Hilbert space $\mathscr{H}$, we denote by π_U the canonically associated $*$-representation of $C^*(G)$ in $\mathscr{B}(\mathscr{H})$ (cf. [8]). We recall that if V is another such representation of G, then U is said to be weakly contained in V (resp. weakly equivalent to V) whenever $\ker \pi_V \subseteq \ker \pi_U$ (resp. $\ker \pi_V = \ker \pi_U$).

By regarding G as a discrete group, we may consider λ as a representation of G_d in $L^2(G)$. To avoid confusion, we shall denote this representation by λ°. It is then clear that $C^*(\lambda(G)) = \pi_{\lambda^\circ}(C^*(G_d))$.

In the above terminology, Theorem 1 (and its proof) may be reformulated as

Theorem 1′. *The following statements are equivalent:*

(i) *G is amenable.*

(ii) *The trivial one-dimensional representation of G_d is weakly contained in λ°.*

(iii) *There exists a character on G_d which is weakly contained in λ°.*

As pointed out to us by A. Valette, the next statement may also be added:

(iv) *There exists a finite-dimensional unitary representation of G_d which is weakly contained in λ°.*

Of course, (iii) $\Rightarrow$ (iv) is trivial, while (iv) $\Rightarrow$ (ii) may be deduced from some results of Fell [8, 13.11.3 and 18.9.15]. Alternatively, (iv) $\Rightarrow$ (i) may be derived from the work of Bekka in [2].

The next lemma plays an important role in the proof of Theorem 3 and is recorded here for the convenience of the reader.

Lemma 2. *λ_d is weakly contained in λ°. Further, if G_d is amenable, then λ_d is weakly equivalent to λ°, and $C^*(\lambda(G))$ is $*$-isomorphic to $C^*_r(G_d)$.*

The first part of Lemma 2 is essentially known and follows immediately by combining the argument given in [1, p. 206] with [10, Theorem 1]. The crucial point is the existence of a net of positive definite functions on G_d associated to λ° which converges pointwisely to the characteristic function of the identity element in G_d, a fact which is also established in the proof of [4, Proposition 1]. The second part of Lemma 2, whose last assertion is shown by Zeller-Meier in [21, Corollary 6] (see also [3, 1.13.ii; 9, Proposition 3.4]), follows now from the fact that, if G_d is amenable, then λ_d weakly contains all unitary representations of G_d (cf. [8, 18.3.5 and 18.3.6]).

Theorem 3. *The following statements are equivalent:*

(i) *G_d is amenable.*

(ii) *G is amenable, and $C^*(\lambda(G)) \simeq C^*_r(G_d)$.*

(iii) *$C^*(\lambda(G))$ is nuclear.*

(iv) *$C^*_r(G_d)$ is nuclear.*

Proof. (i) $\Leftrightarrow$ (iv) is proved by Lance in [13, Theorem 4.2].

(i) $\Rightarrow$ (ii): Suppose G_d is amenable. Then G is amenable ([17, Problem 1.12] or [18, Proposition 4.21]) and $C^*(\lambda(G)) \simeq C^*_r(G_d)$ by Lemma 2.

(ii) $\Rightarrow$ (i): Suppose G is amenable and $C^*(\lambda(G)) \simeq C_r^*(G_d)$. From Theorem 1, we then know that $C^*(\lambda(G))$ possesses a nonzero multiplicative linear functional and, therefore, that $C_r^*(G_d)$ possesses one too. Since $C_r^*(G_d) = C^*(\lambda_d(G_d))$, Theorem 1 now implies that G_d is amenable.

(iii) $\Rightarrow$ (iv): Suppose $C^*(\lambda(G))$ is nuclear. Since λ_d is weakly contained in λ° by Lemma 2, this implies that $\pi_{\lambda_d}(C^*(G_d)) = C_r^*(G_d)$ is a quotient C^*-algebra of $\pi_{\lambda^\circ}(C^*(G_d)) = C^*(\lambda(G))$. As it is known that a quotient C^*-algebra of a nuclear C^*-algebra is itself nuclear [7, Corollary 4], we obtain that $C_r^*(G_d)$ is nuclear.

(iv) $\Rightarrow$ (iii): Suppose $C_r^*(G_d)$ is nuclear. Since we now know that (iv) $\Rightarrow$ (ii), we have $C^*(\lambda(G)) \simeq C_r^*(G_d)$, so $C^*(\lambda(G))$ is nuclear too. $\square$

We conclude this note with some remarks on

$$X(G) = \{\varphi\colon C^*(\lambda(G)) \to \mathbf{C} | \varphi \text{ is nonzero, linear, and multiplicative}\},$$

which is a weak*-closed subset of the state space of $C^*(\lambda(G))$. Theorem 1 says that $X(G) \neq \varnothing$ if and only if G is amenable. When G is abelian, the result of Kodaira and Kakutani mentioned in the introduction may be interpreted as the fact that $X(G)$ is homeomorphic to $\widehat{G}_d$. In the nonabelian case, $X(G)$ is of course a rather primitive C^*-algebra invariant for $C^*(\lambda(G))$, but it has the advantage of being easily computed in some cases, as the following illustrates.

Let H denote a discrete group and CH its commutator subgroup. Then H/CH is abelian, and it is not difficult to show, as it has been observed by Watatani in [20], that if H is amenable then $X(H)$ is homeomorphic to $\widehat{H/CH}$. Hence, if G_d is amenable, via Theorem 2 we get that $X(G)$ is homeomorphic to $\widehat{G_d/CG_d}$. If G is amenable but G_d is not, one can show that $X(G)$ contains a copy of $\widehat{G/\overline{CG}}$ and may itself be embedded as a subgroup of $\widehat{G_d/CG_d}$, but we do not know whether anything more general can be said here. If, for example, $G = \mathrm{SO}(3)$, then $CG_d = G_d$, so $X(G) = \{\hat{1}\}$ (where $\hat{1}$ denotes the state on $C^*(\lambda(G))$ determined by $\hat{1}(\lambda(g)) = 1$ for all g in G; cf. the proof of Theorem 1).

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