A NOTE ON SUBCONTINUA OF $\beta[0, \infty) - [0, \infty)$

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Abstract. Let $M = \bigoplus_{n \in \omega} I_n$ be the topological sum of countably many copies of the unit interval $I$. For any ultrafilter $u \in \omega^*$, we let $M^u = \cap \{\text{cl}_{\beta M}(\bigcup \{I_n : n \in A\}) : A \in u\}$. It is well known that $M^u$ is a decomposable continuum with a very nice internal structure. In this paper, we show:

1. every nondegenerate subcontinuum of $\beta[0, \infty) - [0, \infty)$ contains a copy of $M^u$ for some $u \in \omega^*$.

2. there is no nontrivial simple point in Laver's model for the Borel conjecture.

The second answers a question posed by Baldwin and Smith negatively.

0. Introduction

In this paper we study subcontinua of the Stone-Cech compactification of the nonnegative reals. We refer to [9] and [12] for background on this topic. The unit interval $[0, 1]$ is denoted by $I$. Let $I_n = I \times \{n\}$ for $n \in \omega$, and let $M = \bigoplus_{n \in \omega} I_n$ be the topological sum. For any ultrafilter $u \in \omega^*$, we let

$$M^u = \cap \{\text{cl}_{\beta M}(\bigcup \{I_n : n \in A\}) : A \in u\}.$$ 

It is not difficult to prove that $M^u$ is a continuum (see, e.g., [5]). If we let $i : M \to \omega$ be the map defined by $i(r) = n$ for any $r \in I_n$ and $\beta i : \beta M \to \beta \omega$ be the extension of $i$, it is easy to see that $M^u = \beta i^{-1}(u)$. So every subcontinuum of $\beta M - M$, and therefore, every proper subcontinuum of $\beta[0, \infty) - [0, \infty)$, can be embedded into $M^u$ for some $u \in \omega^*$. Moreover, we have

Theorem 1. Every nondegenerate subcontinuum of $\beta[0, \infty) - [0, \infty)$ contains a copy of $M^u$ for some $u \in \omega^*$.

For any map $f \in \omega I$ and $u \in \omega^*$, let $f^u = \{F \subseteq M : F$ is closed and $\{n : (f(n), n) \in F\} \in u\}$ and $P^u = \{f^u : f \in \omega I\}$. It is well known that $f^u$ is a cut point of $M^u$ if $\{n \in \omega : f(n) \neq 0, 1\} \in u$ [9, (1)]. It is also well known that there are many indecomposable subcontinua with cardinalities $2^c$ in $M^u$ for any $u \in \omega^*$ [9, (19)]. Therefore, by Theorem 1, we have
Corollary. (a) Every subcontinuum of $\beta[0, \infty) - [0, \infty)$ contains an indecomposable subcontinuum.

(b) $\beta[0, \infty)$ does not contain a nondegenerate hereditarily indecomposable subcontinuum.

Assertion (a) is due to Bellamy [2]. (b) was proved by Smith in [10] (van Douwen also announced it in [4]). The following problem was first posed by van Douwen (see the remarks at the end of [11]).

Question 1 (van Douwen). Is there any cut point of $M^u$ which is not in $P^u$?

Definition 1. A point $x \in \beta M$ is said to be (nontrivial) simple if for any $F \in x$ there is $U \in x$ such that $U \subseteq F$ and $U \cap I = \emptyset$ or $U \cap I$ is a (nondegenerate) interval.

Fact 1. (a) [12, §1, Corollary] If $x$ is a cut point of $M^u$ and $x \notin P^u$, then $x$ is a far point of $\beta M$.

(b) [12, Theorem 1.1] $x \in M^u$ is a nontrivial simple point if and only if $x$ is a cut point of $M^u$ and remote point of $\beta M$.

The author [12] proved under CH that there is $u \in \omega^*$ such that there is a cut point of $M^u$ which is not simple. Baldwin and Smith [1] proved that $MA_{\text{countable}}$ implies that there is a nontrivial simple point. They asked

Question 2 (Baldwin and Smith [1]). Is there any nontrivial simple point in ZFC?

Theorem 2. There is no nontrivial simple point in Laver's model for the Borel conjecture.

We would like to mention that the conclusion in Theorem 2 holds in any extension of the model of ZFC + CH by $\omega_2$-iteration with countable support of nontrivial proper forcing notions which add dominating reals and have the Laver property (see [6]).

Question 1 remains open!

1. Proof of Theorem 1

Let $X = [0, \infty)$ and $K \subseteq \beta X - X$ be a nondegenerate subcontinuum. The following lemma was proved by Smith in [10] for locally compact, locally connected metric spaces. We give a direct proof here.

Lemma 1.1. Let $\{U_0, U_1, \ldots, U_m\}$ be a finite open cover of $K$ in $\beta X$ such that $U_i \cap K \neq \emptyset$ for all $i \leq m$. Then there is a closed interval $H \subseteq X$ such that $H \cap U_i \neq \emptyset$ for $i \leq m$ and $H \subseteq \bigcup\{U_i : i \leq m\}$.

Proof. Let $V = \bigcup\{U_0, U_1, \ldots, U_m\}$ and $V' = V \cap X$. Then there are disjoint open intervals $\{J_n : n \in \omega\}$ so that $V' = \bigcup\{J_n : n \in \omega\}$. Let $A_0 = \{n \in \omega : J_n \cap U_0 \neq \emptyset\}$, $V_0 = \bigcup\{J_n : n \in A_0\}$, and $W_0 = \bigcup\{J_n : n \notin A_0\}$. We have $K \subset V \subset (\text{cl}_{\beta X} V_0) \cup (\text{cl}_{\beta X} W_0)$ and $(\text{cl}_{\beta X} V_0) \cap (\text{cl}_{\beta X} W_0) \subset (\text{cl}_{\beta X} V_0) \cap (\text{cl}_{\beta X} W_0) = \text{cl}_{\beta X}(V_0 \cap W_0)$, where $V_0$ and $W_0$ are the closures of $V_0$ and $W_0$ in $X$ respectively. Since $V$ is an open neighbourhood of $K$, we have $K \cap (\text{cl}_{\beta X}(V_0 \cap W_0)) = \emptyset$; therefore, $K \subset \text{cl}_{\beta X} V_0$ since $K$ is connected and $K \cap (\text{cl}_{\beta X} V_0) \supset K \cap U_0 \neq \emptyset$. 

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If we let \( A_i = \{ n \in \omega : J_n \cap U_i \neq \emptyset \text{ for } j \leq i \} \) and \( V_i = \bigcup \{ J_n : n \in A_i \} \) for \( i \leq m \), we can easily show by induction that \( K \subset \text{cl}_{\beta X} V_i \) for \( i \leq m \); so \( A_m \neq \emptyset \). This completes the proof of Lemma 1.1.

We let \( U_0 \) and \( U_1 \) be disjoint open sets of \( \beta X \) so that \( (\text{cl}_{\beta X} U_0) \cap (\text{cl}_{\beta X} U_1) = \emptyset \) and \( U_i \cap K \neq \emptyset \) \( (i = 0, 1) \). Let \( \mathcal{B} \) be the collection of closed intervals so that an interval \([a, b]\) belongs to \( \mathcal{B} \) if and only if the following conditions hold:

1. \([a, b] \cap (U_0 \cup U_1) = \emptyset \) and \( a \neq b \),
2. \([a, b] \subset \text{Br}(U_0 \cap X) \cup \text{Br}(U_1 \cap X) \) and \( a \in \text{Br}(U_0 \cap X) \) if and only if \( b \in \text{Br}(U_1 \cap X) \),

where \( \text{Br} \) denotes the boundary operation in \( X \). Since \( \text{cl}_{\beta X} U_0 \) and \( \text{cl}_{\beta X} U_1 \) are disjoint, \( \mathcal{B} \) is discrete. We enumerate \( \mathcal{B} \) as \( \{ J_n : n \in \omega \} \). We need only show that there is \( u \in \omega^* \) such that \( \bigcap \{ \text{cl}_{\beta X}(\bigcup \{ J_n : n \in A \}) : A \in u \} = K \). Let \( \mathcal{U} \) be an open neighbourhood base of \( K \) in \( \beta X \). For \( U \in \mathcal{U} \), we let \( A_U = \{ n \in \omega : J_n \subset U \} \). Then \( A_U \neq \emptyset \) for \( U \in \mathcal{U} \), since the \( H \) given by Lemma 1.1 for the cover \( \{ U, U_0, U_1 \} \) clearly contains a \( J_n \) for some \( n \in A_U \). Since \( A_U \subset A_V \) for \( U \subset V \) and \( U, V \in \mathcal{U} \), \( \{ A_U : U \in \mathcal{U} \} \) has finite intersection property. Let

\[
M_{\mathcal{U}} = \bigcap \{ \text{cl}_{\beta X}(\bigcup \{ J_n : n \in A_U \}) : U \in \mathcal{U} \}.
\]

It is immediate that \( M_{\mathcal{U}} \subset \bigcap \{ \text{cl}_{\beta X} U : U \in \mathcal{U} \} = K \). Note that

\[
\text{cl}_{\beta X}(\bigcup \{ J_i : i \leq n \}) \cap K = \emptyset \quad \text{for } n \in \omega.
\]

So if \( u \) is an ultrafilter on \( \omega \) and \( \{ A_U : U \in \mathcal{U} \} \subset u \), then \( u \in \omega^* \) and

\[
\bigcap \{ \text{cl}_{\beta X}(\bigcup \{ J_n : n \in A \}) : A \in u \} = K.
\]

This completes the proof of Theorem 1.

2. Proof of Theorem 2

Recall that there is a natural partial order \( \preceq_u \) on \( M^u \) for \( u \in \omega^* \) defined as follows: \( x \preceq_u y \) if and only if there are \( F \in x \) and \( H \in y \) such that \( \{ n \in \omega : F \cap I_n < H \cap I_n \} \subset u \), where \( F \cap I_n < H \cap I_n \) means that \( r < s \) for all \( (r, n) \in F \cap I_n \) and \( (s, n) \in H \cap I_n \). It is easily seen that \( (P^u, \preceq_u) \) is a linearly ordered set. In fact, \( (P^u, \preceq_u) \) is isomorphic to the ultrapower \( (\omega I/u, \preceq_u) \). Moreover, if \( x, y \in M^u \) and \( x \preceq_u y \), then there is a \( p \in P^u \) such that \( x \preceq_u p \preceq_u y \). For \( f, g \in \omega I \) and \( A \in u \), we let

\[
[f, g ; A] = \bigcup \{ [f(n), g(n)] \times \{ n \} : n \in A \}
\]

and

\[
[f, g ; u] = \bigcap \{ \text{cl}_{\beta M}( [f, g ; A] ) : A \in u \}.
\]

It is easily seen that \( M^u = [0, 1 ; u] \), where \( 0(n) = 0 \) and \( 1(n) = 1 \) for \( n \in \omega \). \( [f, g ; u] \) is a continuum since it is homeomorphic to \( M^u \) if \( f^u \preceq_u g^u \). For \( x \in M^u \), we let

\[
[x]_u = \{ y \in M^u : y \text{ and } x \text{ are } <_u \text{-incomparable, or } x = y \}.
\]
Note that \([f, g; u] = \{y \in M^u : f^u \leq y \leq g^u\}\) for \(f, g \in \omega I\), therefore, we have that for \(x \in M^u\)

\[
[x]_u = \bigcap \{[f, g; u] : f^u \leq x \leq g^u\}.
\]

We can also prove that

\[
[x]_u = \left(\text{cl}_{\beta M} \left(\bigcup \{[f, g; u] : f^u \leq g^u < u x\}\right)\right) \cap \left(\text{cl}_{\beta M} \left(\bigcup \{[f, g; u] : x < u f^u \leq g^u\}\right)\right)
\]

(see [9, (16); 12, Lemma 1.2]).

**Lemma 2.1.** (a) A point \(x \in M^u\) is a nontrivial simple point if and only if \([f, g; A] : f, g \in \omega I, A \in u, \text{ and } f^u < u x < u g^u\) is a filter base for \(x\).

(b) A point \(x \in M^u\) is a cut point if and only if \([f, g; A] : f, g \in \omega I, A \in u, \text{ and } f^u < u x < u g^u\) is a neighbourhood base for \(x\).

**Proof.** (a) is obvious. For (b), we assume that \(0^u < u x < 1^u\). It follows easily from (1) that \([x]_u = \{x\}\) if and only if \([	ext{cl}_{\beta M}([f, g; A]) : f, g \in \omega I, A \in u, \text{ and } f^u < u x < u g^u\] is a neighbourhood base of \(x\). Note that in (2) \(\text{cl}_{\beta M} \left(\bigcup \{[f, g; u] : f^u \leq x < u g^u\}\right)\) and \(\text{cl}_{\beta M} \left(\bigcup \{[f, g; u] : f^u \leq g^u \leq g^u\}\right)\) are connected. Therefore, if \([x]_u = \{x\}\), \(x\) is a cut point of \(M^u\); if \([x]_u \neq \{x\}\), then \(x\) is not a cut point of \(M^u\), since \(M^u \setminus \{x\}\) is connected.

For any \(C \subseteq \omega I\) and \(u \in \omega^\ast\), we let \(C^u = \{f^u \in P^u : f \in C\}\). We say that a pair \(E = (C, D)\) of subsets of \(\omega I\) determines a cut point \(x\) in \(M^u\) for some \(u \in \omega^\ast\) if the following two conditions hold:

1. \(C^u < u D^u\), i.e., \(f^u < u g^u\) for all \(f \in C\) and \(g \in D\).
2. \(\text{cl}_{\beta M}([f, g; A]) : f \in C, g \in D, A \in u, \text{ and } f^u < u x < u g^u\) is a neighbourhood base for \(x\).

It is easily seen that for a cut point \(x\) of \(M^u\), if \(C' \subseteq C\) and \(D' \subseteq D\) satisfy that \(C'^u\) is cofinal in \(C^u\) and \(D'^u\) is coinitial in \(D^u\), then \((C, D)\) determines \(x\) if and only if \((C', D')\) determines \(x\). Note that Lemma 2.1(b) says that every cut point of \(M^u\) is determined by a pair of subsets of \(\omega I\).

**Lemma 2.2.** Let \(M \subseteq N\) be transitive models of ZFC such that there is \(r \in \omega^\omega \cap M\) dominating every \(h \in \omega^\omega \cap M\), i.e., \(h(n) < r(n)\) for all but finitely many \(n \in \omega\). If \(u\) is a non-principle ultrafilter on \(\omega\) in \(N\), then no cut point of \(M^u\) is determined by a pair of subsets of \(\omega I\) in \(N\).

**Proof.** Let \(\mathcal{P} = \bigcup_{n \in \omega} \mathcal{P}_n\) be a collection of closed rational subintervals of the unit interval \(I\) such that \(\mathcal{P}_n\) is finite, pairwise disjoint and, for any interval \(J \subseteq I\), if the length of \(J\) is larger than \(1/n\), then \(\{H \in \mathcal{P}_n : H \subseteq J\} \geq 2\).

For \(n \in \omega\), we enumerate \(\mathcal{P}_n\) as \(\{H_{n,j} : j < m_n\}\) so that \(H_{n,j+1} < H_{n,j}\), i.e., the right end of \(H_{n,j-1}\) is less than the left end of \(H_{n,j}\), for \(j < m_n\). Let

\[
F_i = \bigcup \{H_{n,2j+i} : n \in \omega \text{ and } 2j + i < m_n\}
\]

for \(i = 0, 1\). Then \(F_0 \cap F_1 = \emptyset\). Suppose that \(E = (C, D)\) is a pair of subsets of \(\omega I\) in \(M\). For \((f, g) \in C \times D\), we define \(h_{f, g} : \omega \to \omega\) by

\[
h_{f, g}(n) = \begin{cases} 
\min \{m : |f(n) - g(n)| > 1/m\} & \text{if } f(n) \neq g(n), \\
0 & \text{otherwise}.
\end{cases}
\]
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Then $h_{f,g} \in \mathcal{M}$. If $C^u < D^u$, then for $A \in u$ and $i = 0, 1$ we take $[f, g; A] \cap F_i \neq \emptyset$, since $h_{f,g}(n) < r(n)$ for all but finitely many $n \in \omega$. Therefore, no cut point of $M^u$ is determined by $\mathcal{C}$.

Let $\mathbb{P}_{\omega_2}$ be the $\omega_2$-iteration of the Laver forcing with countable support, and let $\mathbb{G}_{\omega_2}$ be $\mathbb{P}_{\omega_2}$-generic over $V$. We assume that the continuum hypothesis holds in $V$. It is well known that a Laver real dominates every real in the ground model; therefore, by [3, Lemma 5.10; 7, Lemma 11], we have

**Corollary 2.3.** There is no cut point of $M^u$ determined by a pair of subsets of $\omega I$ with cardinalities $\omega_1$ in $V[\mathbb{G}_{\omega_2}]$ for any $u \in \omega^*$. The following property of the Laver forcing is weaker than the Laver property in [6]. It is well known that both the Laver forcing and the Mathias forcing have the Laver property and the Laver property is preserved under countable support iterated forcing. We refer to [6] for details.

**Lemma 2.4** [6]. For every $\mathbb{P}_{\omega_2}$-name $\dot{f}$ for a function from $\omega$ to $\omega$ and for every $p \in \mathbb{P}_{\omega_2}$, if $p \Vdash "\dot{f}(n) < 2^n"$ then there exist an extension $q$ of $p$ and a sequence $\{F_n : n \in \omega\}$ of finite subsets of $\omega$ in $V$ such that $q \Vdash "\dot{f}(n) \in F_n$ and $|F_n| < 2^{n_2}$ for $n \in \omega$.

**Lemma 2.5.** Suppose that $p \Vdash_{\mathbb{P}_{\omega_2}} "\dot{f} : \omega \rightarrow I"$. There are an extension $q$ of $p$ and a sequence $\{c_n : n \in \omega\}$ of codes for closed nowhere dense sets in $V$ such that $q \Vdash_{\mathbb{P}_{\omega_2}} "\dot{f}(n) \in c_n for n \in \omega"$.

**Proof.** We modify Miller's argument in [7, §6]. As usual, we deal with the Cantor space $2^\omega$, with the product topology, and the product measure $\mu$ on it. We work in $V[\mathbb{G}_{\omega_2}]$. Assume that $f : \omega \rightarrow 2^\omega$ is a sequence of reals. For $n \in \omega$, we define $\pi_n : (2^\omega)^{\omega} \rightarrow 2^\omega$ by $\pi_n((x_n)) = x_n$. Let $X = \{x \in (2^\omega)^{\omega} : \forall n \in \omega(x(n) < 2^{n_3})\}$. We fix the bijection $\varepsilon : \omega \times \omega \rightarrow \omega$ such that $\varepsilon(n, m+1) < \varepsilon(n, m)$ for $n, m \in \omega$. Define $\theta : X \rightarrow (2^\omega)^\omega$ by

$$\theta(x)(n) = \varepsilon(\varepsilon(n, 0))^{-1} - x(\varepsilon(n, 1))^{-1} \cdots,$$

where we identify $2^{n_3}$ with sequences of 0's and 1's of length $n_3$. By Lemma 2.4 there is a sequence $\{F_n : n \in \omega\}$ of finite subsets of $\omega$ in $V$ such that $\theta^{-1}(f)(n) \in F_n$ for $n \in \omega$. Let $C = \{x \in X : \forall n \in \omega(x(n) \in F_n)\}$. Then $f(n) \in \pi_n(\theta(C))$ for $n \in \omega$. It is easily seen that $\pi_n(\theta(C))$ is a closed subset of $2^\omega$ and

$$\mu(\pi_n(\theta(C))) = \lim_{m \rightarrow \infty} \frac{|F_{e(n,m)}|}{2^{e(n,m)^3}} = \lim_{m \rightarrow \infty} \frac{1}{2^{e(n,m)^2}} = 0.$$ 

This completes the proof since every closed measure zero set is nowhere dense.

**Corollary 2.5.** Let $x \in M^u$ be a nontrivial simple point and $\mathcal{C} = (C, D)$ a pair of subsets of $\omega I$ determining $x$. Then in $V[\mathbb{G}_{\omega_2}]$, for any $u' \in \omega^*$ and $u \subset u'$, there is no $h \in \omega I$ such that $C^u < \{h^u\} < B^u$.

**Proof.** Let $h \in \omega I \cap V[\mathbb{G}_{\omega_2}]$. Then by Lemma 2.4 there is a sequence $\{c_n : n \in \omega\}$ of codes for closed nowhere dense sets of $\omega$ such that $h(n) \in \text{eval}(c_n)$ for $n \in \omega$. Working in $V$, let $F = \bigcup\{\text{eval}^V(c_n) \times \{n\} : n \in \omega\}$. Then $F$ is nowhere dense in $M$. Therefore, there are $f \in C$, $g \in D$, and $A \in u$ such...
that \([f, g; A] \cap F = \varnothing\), since \(x\) is a nontrivial simple point and is determined by \(E\). So in \(V[G_{\omega_2}]\), \(h^{u'} <_{u'} f^{u'}\) or \(g^{u'} <_{u'} h^{u'}\) for any \(u' \in \omega^*\) and \(u \subset u'\).

Now we are in a position to complete the proof of Theorem 2. Suppose there is a nontrivial simple point \(x \in M^{u'} \) in \(V[G_{\omega_2}]\). Then there is a pair \(E = (C, D)\) of subsets of \(\omega I\) determining \(x\). By Lemma 5.10 in [3] there is \(\alpha < \omega_2\) such that, in \(V[G_\alpha]\), \(x'\) is a nontrivial simple point of \(M^{u'}\) and \(E' = (C', D')\) determines \(x'\), where \(x' = x \cap V[G_\alpha]\), \(u' = u \cap V[G_\alpha]\), \(C' = C \cap V[G_\alpha]\), and \(D' = D \cap V[G_\alpha]\). By [7, Lemma 11, Corollary 2.5], \(C^{u'}\) is cofinal in \(C^{u}\) and \(D^{u}\) is coinitial in \(D^{u'}\). Therefore, \(E'\) determines \(x\) in \(V[G_{\omega_2}]\). This is impossible by Corollary 2.3.

**Note added in proof**

We refer to the author’s Continua in \(R^*\), Topology Appl. 50 (1993), 183–197, for more information. In particular, a consistent answer to Question 1 has been given by A. Dow and K. P. Hart.

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