ORTHOCOMPACTNESS IN INFINITE PRODUCT SPACES

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

Abstract. In this paper, we prove the following results for an infinite product space $X = \prod_{\alpha \in K} X_\alpha$.

1. If a dense subspace of $X$ is orthocompact, then it is $\kappa$-metacompact.
2. Assume that all finite subproducts of $X$ are hereditarily orthocompact. If a subspace of $X$ is $\kappa$-metacompact, then it is orthocompact.

1. Introduction

Throughout this paper, all spaces are assumed to be regular $T_1$, $\kappa$ denotes an infinite cardinal, and all product spaces are infinite product spaces. Whenever we consider a product space $\prod_{\alpha \in K} X_\alpha$, we always assume that each $X_\alpha$ contains at least two points.

Beslagić [Be] proved that a product space $X = \prod_{\alpha \in K} X_\alpha$ is $\kappa$-paracompact if it is normal. Conversely, Aoki [Ao] proved that a product space $X$ is normal (orthocompact) if it is $\kappa$-paracompact and each finite subproduct of $X$ is normal (orthocompact). In these connections, we recall Scott’s result [Sl] that a space $Z$ is $\kappa$-metacompact if $Z \times 2^\kappa$ is orthocompact.

In this paper, we prove that a dense subspace of a product space $X = \prod_{\alpha \in K} X_\alpha$ is $\kappa$-metacompact if it is orthocompact. Conversely, we also prove that a product space $X$ is orthocompact if it is $\kappa$-metacompact and each finite subproduct of $X$ is hereditarily orthocompact. Moreover, we can give various applications of these results.

In the rest of this section, we state notation and basic facts. For a set $S$ and a cardinal $\lambda$, we define $[S]^{<\lambda} = \{T \subset S : |T| < \lambda\}$, $[S]^{\leq \lambda} = \{T \subset S : |T| \leq \lambda\}$, and $[S]^\lambda = \{T \subset S : |T| = \lambda\}$, where $|T|$ denotes the cardinality of $T$. Let $\mathcal{U}$ be a collection of subsets of $S$ and $x \in S$. Then $\langle \mathcal{U} \rangle_x$ denotes $\{U \in \mathcal{U} : x \in U\}$. We say that a collection $\mathcal{V}$ of subsets of $S$ is a weak refinement of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. Furthermore, such a $\mathcal{V}$ is a refinement of $\mathcal{U}$ if $\bigcup \mathcal{V} = \bigcup \mathcal{U}$.
Let $X = \prod_{\alpha \in \kappa} X_\alpha$ be a product space. For an $F \subseteq \kappa$, we denote by $X(F)$ the subproduct $\prod_{\alpha \in F} X_\alpha$ and denote by $\pi_F$ the canonical projection map $X \to X(F)$. Such an $X(F)$ is called a finite subproduct of $X$ if $F$ is finite. In particular, we write $X_\alpha$ and $\pi_\alpha$ for $X(\{\alpha\})$ and $\pi(\{\alpha\})$, respectively.

A space $X$ is $(\kappa)$-metacompact if each open cover of $X$ (with cardinality \leq \kappa) has a point-finite open refinement. A space $X$ is (weakly) submetacompact (or (weakly) $\theta$-refinable) if for each open cover $\mathcal{U}$ of $X$ there is a sequence $\{\mathcal{U}_n: n \in \omega\}$ of (weak) open refinements of $\mathcal{U}$ such that for each $x \in X$ there is an $n \in \omega$ such that $(x \in \bigcup \mathcal{U}_n$ and) $\mathcal{U}_n$ is point-finite at $x$. We define (weak) $\kappa$-submetacompactness analogously. In particular, a sequence $\{\mathcal{U}_n: n \in \omega\}$ of covers of $X$ is called a $\theta$-sequence if for each $x \in X$ there is an $n \in \omega$ such that $\mathcal{U}_n$ is point-finite at $x$.

An open cover $\mathcal{U}$ of a space $X$ is interior-preserving if $\bigcap \mathcal{U}'$ is open in $X$ for each $\mathcal{U} \subseteq \mathcal{U}$. A space $X$ is $(\kappa)$-orthocompact if every open cover of $X$ (with cardinality \leq \kappa) has an interior-preserving open refinement. Note that a space $X$ is orthocompact if and only if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcap(\mathcal{V}_x)$ is a (an open) neighborhood of $x$. A space $X$ is (weakly) suborthocompact [KY, Ya] if for each open cover $\mathcal{U}$ of $X$ there is a sequence, $\{\mathcal{U}_n: n \in \omega\}$ of (weak) open refinements of $\mathcal{U}$ such that for each $x \in X$ there is an $n \in \omega$ such that $\bigcap(\mathcal{U}_n)_x$ is a neighborhood of $x$. We define (weak) $\kappa$-suborthocompactness analogously. In particular, a sequence $\{\mathcal{U}_n: n \in \omega\}$ of covers of $X$ is called an $\iota$-sequence [KY] if for each $x \in X$ there is an $n \in \omega$ such that $\bigcap(\mathcal{U}_n)_x$ is a neighborhood of $x$. Clearly, each $\theta$-sequence of open covers of $X$ is an $\iota$-sequence.

By these definitions, the following diagram is easily verified. But note that the ordinal space $\omega_1$ is (hereditarily) orthocompact but not weakly ($\omega_1$-) submetacompact.

\[
\begin{array}{ccc}
\text{metacompact} & \longrightarrow & \text{submetacompact} \\
\downarrow & & \downarrow \\
\text{orthocompact} & \longrightarrow & \text{suborthocompact}
\end{array}
\]

\begin{align*}
\text{weakly submetacompact} & \quad \text{weakly suborthocompact}
\end{align*}

2. $\kappa$-ORTHOCOMPACTNESS IN PRODUCT SPACES

**Theorem 2.1.** Let $Y$ be a dense subspace of a product space $X = \prod_{\alpha \in \kappa} X_\alpha$. Then $Y$ is $\kappa$-metacompact ($\kappa$-submetacompact, weakly $\kappa$-submetacompact) if and only if it is $\kappa$-orthocompact ($\kappa$-suborthocompact, weakly $\kappa$-suborthocompact).

**Proof.** We prove the “if” part. Let $\mathcal{U} = \{U_\alpha: \alpha \in \kappa\}$ be an open cover of $Y$ with cardinality \leq $\kappa$. First we show that $\mathcal{U}$ has an open refinement $(\mathcal{V})$ of cardinality \leq $\kappa$ such that $\text{int}_Y(\bigcap \mathcal{V}_x) = 0$ for each $\mathcal{V}_x \in [\mathcal{W}]^\omega$.

For each $\alpha \in \kappa$, pick distinct two points $p_\alpha(0)$ and $p_\alpha(1)$ in $X_\alpha$. Since $X_\alpha$ is regular $T_1$, we take an open neighborhood $N_\alpha(i)$ of $p_\alpha(i)$, where $i \in \mathbb{Z} = \{0, 1\}$, such that $X_\alpha = N_\alpha(0) \cup N_\alpha(1)$ and $p_\alpha(1 - i) \notin \text{cl}_{X_\alpha} N_\alpha(i)$ for each $\alpha \in \kappa$ and each $i \in \mathbb{Z}$. Let $G_\alpha(i) = \pi_\alpha^{-1}(N_\alpha(i)) \cap Y$ for each $\alpha \in \kappa$ and each $i \in \mathbb{Z}$. Note that each $G_\alpha(i)$ is open in $Y$ and $Y = G_\alpha(0) \cup G_\alpha(1)$ for each $\alpha \in \kappa$.

**Claim.** $\text{int}_Y(\bigcap_{\alpha \in A} G_\alpha(i)) = 0$ for each $A \in [\kappa]^\omega$ and each $i \in \mathbb{Z}$. 

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Proof. Assume that there are an \( A \in [\kappa]^\omega \), an \( i \in 2 \), and a \( y \in Y \) such that \( y \in \text{int}_Y(\bigcap_{\alpha \in A} G_\alpha(i)) \). Then there are an \( F \in [\kappa]^{<\omega} \) and an open set \( V \) in \( X(F) \) such that \( y \in \pi_F^{-1}(V) \cap Y \subset \bigcap_{\alpha \in A} G_\alpha(i) \). Since \( F \) is finite and \( A \) is infinite, pick a \( \beta \) in \( A - F \). Since \( Y \) is dense in \( X \) and \( \beta \notin F \), there is a point \( z \) in \( \pi_F^{-1}(V) \cap \pi_\beta^{-1}(X_\beta - \text{cl} N_\beta(i)) \cap Y \). Then we have \( z \in \pi_F^{-1}(V) \cap Y \subset G_\beta(i) \subset \pi_\beta^{-1}(N_\beta(i)) \). This contradicts \( z \in \pi_\beta^{-1}(X_\beta - \text{cl} N_\beta(i)) \) and completes the proof of the claim.

Put \( \mathcal{U} = \{ U_\alpha \cap G_\alpha(i) : \alpha \in \kappa \text{ and } i \in 2 \} \). Then it follows from the claim that \( \mathcal{U} \) is a desired open refinement of \( \mathcal{U} \).

Now, we prove only the second case. From the above argument, we may assume that \( \mathcal{U} = \{ U_\alpha : \alpha \in \kappa \} \) is an open cover of \( Y \) such that \( \text{int}_Y(\bigcap \mathcal{U}') = 0 \) for each \( \mathcal{U}' \in [\mathcal{U}]^\omega \). There is an \( i \)-sequence \( \{ \mathcal{U}_n : n \in \omega \} \) of open refinements of \( \mathcal{U} \). Then it is easy to see that each \( \mathcal{U}_n \) may be assumed to be a precise open refinement of \( \mathcal{U} \), that is, \( \mathcal{U}_n = \{ V_n(U) : U \in \mathcal{U} \} \) such that \( V_n(U) \subset U \) for each \( U \in \mathcal{U} \). Pick an \( x \in Y \). Choose an \( n \in \omega \) such that \( \bigcap \mathcal{U}_n(x) \) is a neighborhood of \( x \). Assume that \( \bigcap \mathcal{U}_n(x) \) is infinite. Then there is some \( \mathcal{U}' \in [\mathcal{U}]^\omega \) such that \( \{ V_n(U) : U \in \mathcal{U}' \} \subset \bigcap \mathcal{U}_n(x) \). So we have

\[
x \in \text{int}_Y \left( \bigcap \mathcal{U}_n(x) \right) \subset \text{int}_Y \left( \bigcap \{ V_n(U) : U \in \mathcal{U}' \} \right) \subset \text{int}_Y \left( \bigcap \mathcal{U}' \right) = 0.
\]

This is a contradiction. Thus, \( \{ \mathcal{U}_n : n \in \omega \} \) is a \( \theta \)-sequence of open refinements of \( \mathcal{U} \). The proof is complete.

Corollary 2.2. If a product space \( X = \prod_{\alpha \in \kappa} X_\alpha \) is orthocompact (suborthocompact, weakly suborthocompact), then \( X \) is \( \kappa \)-metacompact (\( \kappa \)-submetacompact, weakly \( \kappa \)-submetacompact).

For a space \( X \), \( L(X) \) denotes the Lindelöf degree of \( X \).

Corollary 2.3 [S1, Ya]. A space \( X \) is metacompact (submetacompact, weakly submetacompact) if and only if \( X \times 2^\kappa \) is orthocompact (suborthocompact, weakly suborthocompact) where \( L(X) \leq \kappa \).

Remark. Moreover, we can easily obtain the analogies of [Ao, Theorem 3.1]: A space \( X \) is (weakly) \( \kappa \)-submetacompact if and only if \( X \times A(\kappa) \) is (weakly) \( \kappa \)-suborthocompact, where \( A(\kappa) \) is the one-point compactification of a discrete space of cardinality \( \kappa \). Observe that this is a generalization of Corollary 2.3.

It is known that \( \omega^{\omega_1} \) is not orthocompact; see [Ao, Theorem 3.4] or [S2, Theorem 2.4]. Moreover, we have

Corollary 2.4. \( \omega^{\omega_1} \) is not suborthocompact.

Proof. Assume that \( X = \omega^{\omega_1} \) is suborthocompact. Then, by Corollary 2.2, \( X \) is \( \omega_1 \)-submetacompact. Since the weight of \( X \) is \( \omega_1 \), it is submetacompact. But it follows from the statement in [PP, p. 63] that \( X \) is not submetacompact. This is a contradiction.

Let \( Y \) be a \( \Sigma \)-product of \( \{ X_\alpha : \alpha \in \kappa \} \). Then \( Y \) is said to be proper [Pr, §7] if \( Y \) is a proper subspace of \( \prod_{\alpha \in \kappa} X_\alpha \) (i.e., \( \kappa \geq \omega_1 \) and \( |X_\alpha| \geq 2 \) for each \( \alpha \in \kappa \)).

Corollary 2.5. All proper \( \Sigma \)-products are not weakly suborthocompact.

Proof. Let \( Y \) be a proper \( \Sigma \)-product of \( \{ X_\alpha : \alpha \in \kappa \} \), where \( \kappa \geq \omega_1 \). Assume that \( Y \) is weakly suborthocompact. Since \( Y \) is dense in \( X \), it follows from
Theorem 2.1 that \( Y \) is weakly \( \kappa \)-submetacompact. Since \( Y \) contains a closed subspace which is homeomorphic to the ordinal space \( \omega_1 \) (cf. [Pr, Proposition 7.2]), the space \( \omega_1 \) is weakly \( \kappa \)-submetacompact. But it is well known that the space \( \omega_1 \) is not weakly \( \omega_1 \)-submetacompact. This is a contradiction.

**Corollary 2.6.** Let \( X \) be a product space of paracompact \( p \)-spaces (e.g., metrizable spaces). Then the following are equivalent.

1. \( X \) is (sub)orthocompact.
2. \( X \) is normal.
3. \( X \) is paracompact.

Using Corollary 2.4, the proof is similar to that of [Pr, Corollary 6.5].

**Remark.** The condition “paracompact \( p \)-space” in Corollary 2.6 is essential. In fact, let \( X \) be a \( \Sigma \)-product in \( 2^{\omega_1} \). Then \( X \) is homeomorphic to \( X^{\omega} \). It follows from [Pr, Theorem 7.4] and Corollary 2.5 that \( X^{\omega} \) is normal but not weakly suborthocompact.

We obtain the following generalization of [Ao, Theorem 3.5] or [S2, Theorem 2.5].

**Corollary 2.7.** The following are equivalent for a space \( X \).

1. \( X \) is compact.
2. \( X^{\kappa} \) is suborthocompact for any cardinal \( \kappa \).
3. \( X^{\kappa} \) is suborthocompact for some cardinal \( \kappa \) with \( \kappa \geq \omega_1 \cdot L(X) \).

Using Corollaries 2.3 and 2.4, the proof is parallel to that of [Ao, Theorem 3.5].

If \( \omega^{\omega_1} \) was not weakly submetacompact, then “suborthocompact” in most of our corollaries could be replaced by “weakly suborthocompact”. Hence, we conclude this section with the following problem.

**Problem 2.8.** Is \( \omega^{\omega_1} \) not weakly submetacompact?

### 3. \( \kappa \)-METACOMPACTNESS IN PRODUCT SPACES

As the converse of Corollary 2.2, we obtain the following:

**Theorem 3.1.** Assume that all finite subproducts of a product space \( X = \prod_{\alpha \in \kappa} X_\alpha \) are hereditarily orthocompact. If a subspace \( Y \) of \( X \) is \( \kappa \)-metacompact (\( \kappa \)-submetacompact, weakly \( \kappa \)-submetacompact), then it is orthocompact (suborthocompact, weakly suborthocompact).

**Proof.** We prove only the second case. Let \( \mathcal{U} \) be an open cover of \( Y \). We may assume that, for each \( U \in \mathcal{U} \), there are an \( F(U) \in [\kappa]^{<\omega} \) and an open set \( G(U) \) in \( X(F(U)) \) such that \( U = \pi_{F(U)}^{-1}(G(U)) \cap Y \). For each \( F \in [\kappa]^{<\omega} \), put \( \mathcal{Z}_F = \{ U \in \mathcal{U} : F(U) = F \} \) and \( G_F = \bigcup \{ G(U) : U \in \mathcal{Z}_F \} \). Then it is easy to check that each \( G_F \) is open in \( X(F) \) and \( \mathcal{A} = \{ \pi_F^{-1}(G_F) \cap Y : F \in [\kappa]^{<\omega} \} \) is an open cover of \( Y \). Since each \( X(F) \) is hereditarily orthocompact, there is an interior-preserving collection \( \mathcal{B}(F) = \{ B(F) : U \in \mathcal{Z}_F \} \) of open sets in \( X(F) \) such that \( B(F) \subset G(U) \) for each \( U \in \mathcal{Z}_F \) and \( \bigcup \mathcal{B}(F) = G_F \). By the \( \kappa \)-submetacompactness of \( Y \) and \( |\mathcal{A}| \leq \kappa \), there is a \( \theta \)-sequence \( \{ V_F^n : n \in \omega \} \) of open refinements of \( \mathcal{A} \). We may assume that \( \mathcal{V}_n = \{ V_F^n : F \in [\kappa]^{<\omega} \} \).
such that $V_F^n \subset \pi_F^{-1}(G_F) \cap Y$ for each $F \in [\kappa]^{<\omega}$ and each $n \in \omega$. Put $\mathcal{W}_F^n = \{\pi_F^{-1}(B_F(U)) : U \in \mathcal{U}_F\}$ for each $F \in [\kappa]^{<\omega}$ and each $n \in \omega$. Then it is easy to check that each $\mathcal{W}_F^n$ is an interior-preserving collection of open sets in $Y$ whose union is $V_F^n$. Put $\mathcal{W}_n = \bigcup\{\mathcal{W}_F^n : F \in [\kappa]^{<\omega}\}$ for each $n \in \omega$. Observe that each $\mathcal{W}_n$ is an open refinement of $\mathcal{U}$. We show that $\{\mathcal{W}_n : n \in \omega\}$ is an $\omega$-sequence. Pick an $x \in X$. Since $\{\mathcal{V}_n : n \in \omega\}$ is a $\theta$-sequence, take an $n \in \omega$ such that $(\mathcal{V}_n)_x$ is finite, say $(\mathcal{V}_n)_x = \{V_F^n : F \in \mathcal{F}\}$ for some $\mathcal{F} \subseteq [\kappa]^{<\omega}$. Since $\mathcal{W}_F^n$ is interior-preserving and $x \in V_F^n = \bigcup \mathcal{W}_F^n$, $\bigcap (\mathcal{W}_F^n)_x$ is an open neighborhood of $x$ for each $F \in \mathcal{F}$. Since $(\mathcal{W}_n)_x = \bigcup_{F \in \mathcal{F}} (\mathcal{W}_F^n)_x$ and $|\mathcal{F}| < \omega$, it follows that $\bigcap (\mathcal{W}_n)_x = \bigcap_{F \in \mathcal{F}} (\bigcap (\mathcal{W}_F^n)_x)$ is an open neighborhood of $x$. This completes the proof.

Considering [Ao, Corollary 2.5], it is natural to raise

**Problem 3.2.** If a product space $X = \prod_{\alpha \in \kappa} X_\alpha$ is $\kappa$-metacompact and all finite subproducts of $X$ are orthocompact, is $X$ orthocompact?

**Proposition 3.3.** Assume that all finite subproducts of a product space $X = \prod_{\alpha \in \kappa} X_\alpha$ are hereditarily metacompact. If a dense subspace $Y$ of $X$ is $\kappa$-orthocompact ( $\kappa$-suborthocompact, weakly $\kappa$-suborthocompact), then $Y$ is metacompact (submetacompact, weakly submetacompact).

**Proof.** The second case: Observe that $Y$ is $\kappa$-submetacompact according to Theorem 2.1, because $Y$ is a $\kappa$-suborthocompact dense subspace of $X$. Then replacing “interior-preserving” by “point-finite” in the proof of Theorem 3.1, we can prove similarly.

**Corollary 3.4.** Let $X$ be a product space of metrizable spaces and $Y$ a dense subspace of $X$. Then $Y$ is orthocompact (suborthocompact, weakly suborthocompact) if and only if it is metacompact (submetacompact, weakly submetacompact).

We can consider this is an analogue of [Ba, Theorem 1].

**Remark.** Under the assumption of Corollary 3.4, $X$ is normal if and only if it is paracompact (see Corollary 2.6). But one cannot replace “orthocompact” and “metacompact” by “normal” and “paracompact”, respectively, in Corollary 3.4. In fact, let $Y$ be a $\Sigma$-product in $X = 2^{\omega_1}$. Then $Y$ is a normal nonparacompact, dense subspace of $X$.

**References**


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