AN EXTENSION OF THE HEINZ-KATO THEOREM

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Dedicated to Professor Huzihiro Araki on his sixtieth birthday with respect and affection

Abstract. An extension of the Heinz-Kato theorem is given.

In this paper, we shall extend the famous and well-known Heinz-Kato theorem. Please note that a capital letter means a bounded linear operator on a complex Hilbert space $H$.

Theorem A (Heinz-Kato [1, 2]). Let $T$ be an operator on a Hilbert space $H$. If $A$ and $B$ are positive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in H$, then the following inequality holds for all $x, y \in H$:

$$|(Tx, y)| \leq \|A^\alpha x\| \|B^{1-\alpha} y\| \text{ for any } \alpha \in [0, 1].$$

We shall show an extension of Theorem A as follows.

Theorem 1. Let $T$ be an operator on a Hilbert space $H$. If $A$ and $B$ are positive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in H$, then the following inequality holds for all $x, y \in H$:

$$|(T^{|T|^{\alpha\beta-1}}x, y)| \leq \|A^\alpha x\| \|B^\beta y\| \text{ for any } \alpha, \beta \text{ such that } \alpha, \beta \in [0, 1] \text{ and } \alpha + \beta \geq 1.$$

We remark that Theorem A follows by Theorem 1 putting $\alpha + \beta = 1$ in Theorem 1, so that Theorem 1 can be considered as an extension of the Heinz-Kato theorem.

In order to give a proof of Theorem 1, we need the following Theorem B [1, 3] for which there are a lot of proofs—among them, a nice one given in [4].

Theorem B (Löwner-Heinz). If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ holds for each $\alpha \in [0, 1]$.

Also we cite the following obvious lemma.

Lemma. Let $S$ be positive operator. Then:

(i) $(Sx, x) = 0$ holds for some vector $x$ iff $Sx = 0$.

(ii) $N(S^q) = N(S)$ holds for any positive real number $q$, where $N(X)$ denotes the kernel of an operator $X$.

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Proof of Theorem 1. First of all, the hypothesis \( \|Tx\| \leq \|Ax\| \) for all \( x \in H \) is equivalent to

\[
(3) \quad |T|^2 \leq A^2.
\]

Also the hypothesis \( \|T^*y\| \leq \|By\| \) for all \( y \in H \) is equivalent to

\[
(4) \quad |T^*|^2 \leq B^2.
\]

Applying Theorem B to (3) and (4), for any \( x, y \in H \) we have

\[
(5) \quad (|T|^{2\alpha} x, x) \leq (A^{2\alpha} x, x) \quad \text{for each } \alpha \in [0, 1],
\]

\[
(6) \quad (|T^*|^{2\beta} y, y) \leq (B^{2\beta} y, y) \quad \text{for each } \beta \in [0, 1].
\]

Let \( N(X) \) denote the kernel of an operator \( X \). Let \( T = U|F|^T \) be the polar decomposition of an operator \( T \), where \( U \) is partial isometry and \( |T| = (T^*T)^{1/2} \) and \( N(U) = N(|T|) \).

In the case \( \alpha, \beta \in [0, 1] \) such that \( \beta > 0 \) and \( \alpha + \beta \geq 1 \), we recall the following well-known relation on the polar decomposition of \( T \):

\[
(7) \quad |T|^2 = U|T|^{2\beta} U^* \quad \text{holds for any } \beta > 0.
\]

Then for all \( x, y \in H \) we have

\[
(8) \quad (|T|^{2\alpha-1} x, y)^2 = \|(U|T|^{\alpha+\beta} x, y)^2 = \|(|T|^2 x, |T|^2 U^* y)^2 \leq \|T^{\alpha} x\|^2 \|T^\beta U^* y\|^2 = (|T|^{2\alpha} x, x)(U|T|^{2\beta} U^* y, y)
\]

\[
= (|T|^{2\alpha} x, x)(|T^*|^{2\beta} y, y) \quad \text{(by (7))}
\]

\[
\leq (A^{2\alpha} x, x)(B^{2\beta} y, y) \quad \text{(by (5) and (6))}
\]

for any \( \alpha \) and \( \beta \) such that \( \alpha, \beta \in [0, 1] \) and \( \alpha + \beta \geq 1 \); that is, (2) holds because the result is trivial in the case \( \beta = 0 \).

Hence the proof of Theorem 1 is complete.

Remark 1. In the case \( \alpha > 0 \) and \( \beta > 0 \), the equality in (2) holds for some \( x \) and \( y \) if \( f |T|^{2\alpha} x \) and \( |T|^{\alpha+\beta-1} T^* y \) are linearly dependent and \( |T|^{2\alpha} x = A^{2\alpha} x \) and \( |T^*|^{2\beta} y = B^{2\beta} y \) hold for some \( x \) and \( y \) together.

In fact, in the case \( \alpha > 0 \) and \( \beta > 0 \), the equality in the first inequality of (8) holds if \( f |T|^{\alpha} x \) and \( |T^\beta U^* y \) are linearly dependent, that is, \( |T|^{2\alpha} x \) and \( |T|^{\alpha+\beta-1} |T| U^* y \) are linearly dependent by (ii) of Lemma, namely, \( |T|^{2\alpha} x \) and \( |T|^{\alpha+\beta-1} T^* y \) are linearly dependent.

The equality in the last inequality of (8) holds if the equality of (5) and the equality of (6) hold together, that is, \( |T|^{2\alpha} x = A^{2\alpha} x \) and \( |T^*|^{2\beta} y = B^{2\beta} y \) hold together for some vector \( x \) and \( y \) by (1) of Lemma; so the proof of the equality is complete.

Remark 2. The condition \( \alpha + \beta \geq 1 \) in Theorem 1 is unnecessary if \( T \) is a positive operator or invertible operator. This is easily seen in the proof of Theorem 1.

Remark 3. We remark that a condition for which \( |T|^{2\alpha} x \) and \( |T|^{\alpha+\beta-1} T^* y \) are linearly dependent is equivalent to that \( T|T|^{\alpha+\beta-1} x \) and \( |T^*|^{2\beta} y \) are linearly dependent. In fact, the former condition is equivalent to that \( |T|^{\alpha} x \) and \( |T^\beta U^* y \) are linearly dependent as stated in the proof of the equality in the first inequality of (8), and this condition is equivalent to that \( U|T|^{\alpha+\beta} x \) and \( U|T|^{2\beta} U^* y \) are linearly dependent by (ii) of Lemma and \( N(U) = N(|T|) \), that is, \( T|T|^{\alpha+\beta-1} x \) and \( |T^*|^{2\beta} y \) are linearly dependent by (7).
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REFERENCES


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