

## TIGHT ANALYTIC IMMERSIONS OF HIGHLY CONNECTED MANIFOLDS

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**ABSTRACT.** An immersion of a manifold which minimizes total absolute curvature is called tight. In this paper we determine which  $(k - 1)$ -connected (but not  $k$ -connected) manifolds of dimension  $2k$ , with trivial  $k$ th Stiefel-Whitney class, admit codimension 2 tight analytic immersions.

### 0. INTRODUCTION

A smooth immersion of a compact manifold  $f : M \rightarrow \mathbb{R}^N$  is said to be *tight* if it minimizes the total absolute curvature among all immersions of  $M$ . For the case where  $M$  is a surface, this is equivalent to saying that  $f$  satisfies the two-piece property; i.e., the preimage of a hyperplane in  $\mathbb{R}^N$  decomposes  $M$  into at most two components.

One of the simplest cases that can be considered is that of tight immersions of  $2k$ -dimensional  $(k - 1)$ -connected manifolds in  $\mathbb{R}^{2k+l}$ . Kuiper [5] was the first to study these and showed that they must satisfy rather stringent conditions. He later conjectured [6] that any highly connected smooth compact  $2k$ -dimensional manifold with  $k \geq 2$  and trivial  $k$ th Stiefel-Whitney class, admitting a tight immersion into Euclidean space, is homeomorphic to  $S^k \times S^k$ . Counterexamples to this conjecture were constructed by Hebda [4], who showed that connected sums of arbitrarily many copies of  $S^k \times S^k$  admit smooth tight codimension 1 embeddings. Thorbergsson [9] used the ideas of Hebda to construct codimension 2 substantial embeddings of this connected sum. He showed that, if the  $k$ th Stiefel-Whitney class is trivial,  $\omega_k(M) = 0$ , then  $M^{2k}$  has the same cohomology ring as the connected sum  $(S^k \times S^k) \# \dots \# (S^k \times S^k)$ , and, if  $k > 4$ , then  $M^{2k}$  is diffeomorphic to  $(S^k \times S^k) \# \dots \# (S^k \times S^k) \# \Sigma$ , where  $\Sigma$  is a sphere with some differentiable structure. In general, one can show that  $\omega_k(M) = 0$  for a highly connected  $2k$ -dimensional manifold with  $k \neq 1, 2, 4, \text{ or } 8$ . The geometric interpretation of this condition for highly connected manifolds is that no homology class has self-intersection number 1 mod 2.

In this paper we make the extra assumption that the immersions be analytic and prove that under this restriction Kuiper's conjecture is true.

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We prove the following theorem.

**Theorem.** *Let  $f : M^{2k} \rightarrow \mathbb{R}^{2k+2}$  be a tight, analytic, substantial immersion of a compact  $(k - 1)$ -connected manifold, with  $\omega_k(M) = 0$  if  $k \leq 2$ . If  $M$  is not  $k$ -connected, then  $M$  is homeomorphic to  $S^k \times S^k$ .*

Thorbergsson [9] showed that for substantial tight immersions of compact highly connected manifolds, if  $\omega_k(M^{2k})$  vanishes, then the codimension must be 1 or 2. This corollary then follows.

**Corollary.** *Let  $f : M^{2k} \rightarrow \mathbb{R}^{2k+l}$ ,  $l > 1$ , be a tight, analytic, substantial immersion of a compact  $(k - 1)$ -connected manifold with  $\omega_k(M) = 0$ . If  $M$  is not  $k$ -connected, then  $M$  is homeomorphic to  $S^k \times S^k$ .*

This is a generalization of a result of Thorbergsson [10], who proved this when  $M$  is an orientable surface.

## 1. PRELIMINARIES

This section will cover the basic definitions and some technical results that will be used in later proofs. For more information about tight immersions see [3] or [6].

Given an immersion  $f : M \rightarrow \mathbb{R}^N$  of a compact manifold into Euclidean space, one can examine whether the total absolute curvature of the manifold with respect to the immersion  $f$  attains the minimum possible value. If this is the case, then the immersion is called *tight*. Examining the work of Chern and Lashof [2] results in the following equivalent definition: An immersion is tight if and only if every height function which is a Morse function has the minimal number of critical points required by the Morse inequalities for some field. Notice that almost every height function is a Morse function. The proof of the Morse inequalities leads to the following equivalent definition: An immersion  $f : M \rightarrow \mathbb{R}^N$  of a compact manifold into Euclidean space is said to be *tight*, if for almost every closed half space  $\mathcal{L}$  in  $\mathbb{R}^N$ , the induced homomorphism of singular homology with respect to some field  $\mathbb{F}$

$$H_*(f^{-1}(\mathcal{L}); \mathbb{F}) \rightarrow H_*(M; \mathbb{F})$$

is injective for every  $*$ .

By replacing singular homology theory with Čech theory, the above definition is true for every half space, but, since Čech homology is not a standard theory, we use the following equivalent definition.

**1.1. Definition.** A continuous map  $f : M \rightarrow \mathbb{R}^N$  of a compact connected topological space into Euclidean space is called *tight* if there is a field  $\mathbb{F}$  such that the induced homomorphism in Čech cohomology

$$\check{H}^*(M; \mathbb{F}) \rightarrow \check{H}^*(f^{-1}(\mathcal{L}); \mathbb{F})$$

is surjective for every  $*$ , and every half space  $\mathcal{L}$ .

Although Čech homology is not a standard theory, we will use it occasionally in this paper for convenience, and in general we will try to use Čech cohomology. It should be noted that, for surfaces, this definition is equivalent to the two-piece property. In any case, for any tight immersion, the two-piece property holds.

For a given unit vector  $\xi$ , let  $h_\xi(x) = x \cdot \xi$ . Then, for an immersion  $f : M^n \rightarrow \mathbb{R}^N$ , the half space  $\mathcal{L}$ , defined by  $\mathcal{L} = \{x \in \mathbb{R}^N \mid h_\xi(x) \geq c\}$ , and the hyperplane bounding  $\mathcal{L}$ , given by  $\ell = \{x \in \mathbb{R}^N \mid h_\xi(x) = c\}$ , are said to *support*  $f(M)$  if the boundary but not the interior of  $\mathcal{L}$  has nonempty intersection with  $f(M)$ . If  $f(M)$  is not contained in any hyperplane  $\ell$ , then  $f$  is said to be a *substantial* immersion. The set  $\Psi(\xi) = f^{-1}(\mathcal{L}) = f^{-1}(\ell)$  is then called a *top-set* if  $\mathcal{L}$  supports  $f(M)$ , and if  $\Psi(\xi) \neq M$ , which is always true if  $f$  is substantial. Notice that the function  $h_\xi \circ f$  takes its maximum value on the top-set  $\Psi(\xi)$ , and we call the map  $f|_{\Psi(\xi)}$  a *top-map* of  $f$ . For a pair of orthogonal vectors  $\xi_1$  and  $\xi_2$ , a top-set  $\Psi(\xi_1, \xi_2)$  of a top-map is called a *top<sup>2</sup>-set*. A *top\*-set* can then be defined inductively to be a top-set of a top<sup>\*</sup>-1-set. In [6], Kuiper proved the fundamental result that top\*-maps of tight maps are tight. In general we will also call the image of a top-set  $\Omega = f(\Psi)$  a top-set as well. Any point contained in a top-set is called a *top-point*.

Let  $f : M^n \rightarrow \mathbb{R}^N$  be an immersion with normal bundle  $NM$ . We define the *convex envelope* of  $f$  to be the boundary of the convex hull of  $f(M)$ , and we define a *convex m-cycle* to be the convex envelope of a compact subset that spans an  $(m + 1)$ -dimensional affine space and is therefore homeomorphic to a sphere. Say that  $p$  is a *nondegenerate convex point* of the immersion  $f$  if there is some height function  $h_\xi$  with a nondegenerate maximum at  $f(p)$ . If this is the case, then for all  $X, Y \in T_pM$ ,

$$H_\xi(X, Y) = \langle \xi, \alpha(X, Y) \rangle,$$

where  $H_\xi$  is the Hessian of  $h_\xi$  and  $\alpha$  is the second fundamental form. Since  $p$  is nondegenerate, then  $\alpha(X, X) \neq 0$  for every  $X \neq 0$ , and the set  $\mathcal{A} = \{\alpha(X, X) \mid X \in T_pM\}$  is contained in a half space of  $N_pM$ . Denote the convex hull of  $\mathcal{A}$  by  $\mathcal{K}_p$ . It is easy to see that  $\mathcal{K}_p$  spans  $N_pM$  if  $f$  is a tight substantial immersion and  $p$  is a nondegenerate convex point.

If  $p$  is a nondegenerate convex point of  $f$ , then there is a hyperplane through  $f(p)$  that supports  $f(M)$ , and this hyperplane contains  $T_pM$ . The set of hyperplanes in  $\mathbb{R}^{n+k}$  that contain  $T_pM$  form a  $(k - 1)$ -dimensional space  $\beta$  in the dual projective space  $P^*$  of  $\mathbb{R}P^{n+k} \supset \mathbb{R}^{n+k}$ . Then the supporting hyperplanes at  $p$  form a compact subset  $\mathcal{E}$  of  $\beta$ , which can degenerate to a point. Call the points in  $\partial\mathcal{E}$  the *extremal supporting hyperplanes at  $p$* . In fact, the extremal supporting hyperplanes at  $p$  are precisely those hyperplanes in  $\mathbb{R}^{n+k}$  which contain both  $T_pM$  and the hyperplanes in  $N_pM$  which support  $\mathcal{K}_p$ .

The following lemmas proven in [9] will be important in proofs to follow.

**1.2. Lemma.** *Let  $f : M \rightarrow \mathbb{R}^N$  be a tight immersion. If a height function  $h_\xi$  has a nondegenerate critical point at  $p$  of index  $l$ , then*

$$\check{H}^l(f^{-1}(\mathcal{L}(0)), \mathbb{F}) \neq 0,$$

where  $\mathcal{L}(t) = \{x \in \mathbb{R}^N \mid h_\xi(x - f(p)) \leq -t\}$ .

**1.3. Remark.** Notice that it is only necessary for  $f$  to be a continuous map of a compact space  $M$  into  $\mathbb{R}^N$  and that, in a neighborhood of  $p$ ,  $f$  is smooth. This is true since in the proof of the lemma the fact that

$$\check{H}^*(f^{-1}(\mathcal{L}(0)), f^{-1}(\mathcal{L}(\epsilon))) \neq 0, \quad \text{for } \epsilon > 0,$$

is dependent upon the fact that a Morse chart can be constructed around  $p$ .

1.4. *Remark.* Since  $\check{H}^*(M) \rightarrow \check{H}^*(f^{-1}(\mathcal{L}))$  must be surjective, then if  $M^{2k}$  is highly connected, it is only possible for there to be critical points of index 0,  $k$ , or  $2k$ .

1.5. **Lemma.** Let  $p$  be a nondegenerate convex point of the tight immersion  $f : M \rightarrow \mathbb{R}^N$ . Let  $h \subset N_pM$  be a hyperplane of support of  $\mathcal{K}_p$ , and let  $\xi \in N_pM$  be orthogonal to  $h$  with  $h_\xi(\alpha(X, X)) \geq 0$  for all  $X \in T_pM$ .

(a) Then  $\mathcal{E}_h = \{X \in T_pM \mid \alpha(X, X) \in h\}$  is a linear subspace, and  $\alpha(\mathcal{E}_h, T_pM) \subset h$ .

(b) Let  $\xi(t)$  be a curve in  $N_pM$  such that  $\xi(0) = \xi$  and  $h_{\xi(t)}(\alpha(X, X)) < 0$  for every nonzero  $X \in \mathcal{E}_h$  and  $t > 0$ . Then there is an  $\epsilon > 0$  such that for all  $0 < t \leq \epsilon$  the function  $h_{\xi(t)} \circ f$  has a nondegenerate critical point at  $p$  of index equal to the dimension of  $\mathcal{E}_h$ .

We will review some properties of analytic subsets. For more information see, for example, [1].

1.6. **Definition.** Let  $f : M \rightarrow \mathbb{R}^N$  be an analytic immersion. A set  $\Omega \subset f(M)$  is called an *analytic subset* of  $f(M)$  if for each point  $p \in f(M)$  there is a neighborhood  $U$  containing  $p$  and analytic functions  $f_1, \dots, f_n$  in this neighborhood such that

$$\Omega \cap U = \{x \in U \mid f_1(x) = \dots = f_n(x) = 0\}.$$

1.7. *Remark.* If any open set  $\Omega \subset \mathbb{R}^l$  is an analytic subset in  $\mathbb{R}^l$  then  $\Omega = \mathbb{R}^l$ .

A point  $p \in \Omega$  is called *regular* if there is a neighborhood  $U$  containing  $p$  such that  $\Omega \cap U$  is an open submanifold of  $f(M)$ . The dimension of the set  $\Omega$  at the regular point  $p$  is the dimension of this submanifold and is denoted by  $\dim_p \Omega$ . The set of all regular points is denoted by  $\text{reg } \Omega$ , and every point in the complement  $\Omega \setminus \text{reg } \Omega$  is called a *singular point*. The set of singular points is denoted by  $\text{sng } \Omega$ . An important fact about analytic subsets that we will make use of is that the set of regular points of an arbitrary analytic subset  $\Omega$  is dense in  $\Omega$ ; i.e.,  $\overline{\text{reg } \Omega} = \Omega$ . We also define

$$\underline{\dim}_p \Omega = \lim_{x \rightarrow p} \dim_x \Omega, \quad \underline{\dim} \Omega = \min_{p \in \Omega} \{\underline{\dim}_p \Omega\}.$$

Notice that an analytic subset of an analytic immersion  $f$  cannot contain any straight line segments if  $M$  is compact. We know that, if  $\bar{f}(N)$  is an analytic immersion of a manifold and an analytic subset  $B$  of  $\bar{f}(N)$  contains a nonempty open subset in  $\bar{f}(N)$ , then by the uniqueness of analytic functions  $B = \bar{f}(N)$ . Now suppose that  $\ell$  is a straight line in  $\mathbb{R}^N$ . Then  $\ell \cap f(M)$  is an analytic subset of  $\ell$ , and, if  $\ell \cap f(M)$  contains an open subset of  $\ell$ , then it must be all of  $\ell$ , contradicting the compactness of  $M$ .

Now let  $f : M^{2k} \rightarrow \mathbb{R}^N$  be a tight analytic immersion of a compact manifold. Let  $\Psi \subset M$  be a top-set with respect to some height function  $h_\xi$ , where  $t = 0$  is the minimum of  $h_\xi \circ f$ . Then  $\Omega = f(\Psi)$  is a connected, closed, analytic subset of  $f(M)$ . If  $\Omega$  is not one point, then it is the union of a finite number of regular open  $m$ -dimensional sets,  $(m - 1)$ -dimensional sets,  $\dots$ , arcs, and points, and this union is such that the set  $\Omega$  is closed.

## 2. TOP-SETS OF TIGHT ANALYTIC IMMERSIONS

The following proposition guarantees that on every open set there is a non-degenerate height function with a critical point in that open set.

**2.1. Lemma.** *For every open neighborhood  $W$  of a smooth immersion  $f : M^n \rightarrow \mathbb{R}^N$ , there is a height function with a nondegenerate critical point on  $W$ , or  $f(W)$  contains a straight line segment.*

*Proof.* Let  $A_\xi(p)$  be the shape operator at  $p \in W$  corresponding to the normal direction  $\xi$ . We will show that, if  $\text{rank } A_\xi(p) \leq n - 1$  for all  $p \in W$  and for all  $\xi \in N_p W$ , then  $f(W)$  contains a straight line segment. This is sufficient to prove the proposition, since  $\text{rank } A_\xi(p) \leq n - 1$  if and only if every height function  $h_\xi(f(p))$  is degenerate at  $p$ .

First consider hypersurfaces in  $\mathbb{R}^N$ . If  $\lambda$  is a principal curvature of constant multiplicity  $\nu > 1$ , then the leaves of the principal foliation corresponding to  $\lambda$  are umbilic submanifolds in  $\mathbb{R}^N$ . See, for example, [3, Theorem 4.5]. If we assume that  $\lambda$  is constant on the hypersurface, then the assumption on the multiplicity can be weakened to  $\nu \geq 1$ , since the condition  $\nu > 1$  is only necessary to show that  $\lambda$  is constant on each of the leaves. It also follows that if we assume that  $\lambda \equiv 0$  then, in fact, the leaves of the principal foliation are totally geodesic.

Next, let  $f : M^n \rightarrow \mathbb{R}^N$  be an immersion with unit normal bundle  $N^1$ , projection  $\pi : N^1 \rightarrow M$ , and define  $\bar{f} : N^1 \rightarrow \mathbb{R}^N$  by  $\bar{f}(\xi) = f(\pi\xi) + t\xi$ , where  $t \in \mathbb{R}$  is chosen such that  $\bar{f}$  is an embedding on some open set  $\bar{W} \subset N^1$ . Suppose that  $\text{rank } A_\xi \leq n - 1$  on  $W = \pi\bar{W}$ . Let  $\bar{A}_t$  be the shape operator of  $\bar{f}$ . Then by [3, Theorem 3.2],  $\text{rank } \bar{A}_t = \text{rank } A_\xi + k - 1 \leq n + k - 2$ . Let  $\bar{W}_0$  be an open subset of  $\bar{W}$  on which  $\text{rank } \bar{A}_t$  is maximal. Then there is a principal curvature  $\bar{\lambda} = 0$  of multiplicity  $\bar{\nu} = \text{nullity } \bar{A}_t > 0$  on  $\bar{W}_0$ , so the leaves of  $T_0 = \{X \in T_\xi \bar{W}_0 \mid \bar{A}_t X = 0\}$  are totally geodesic.

Let  $\bar{\gamma}(s)$  be an integral curve of  $T_0$  that embeds as a straight line segment in  $\bar{W}$ . Then  $\gamma(s) = \pi\bar{\gamma}(s)$  is a curve on  $W$ . Let  $\xi = \bar{\gamma}(s_0)$  and  $p = \pi\xi$ . Then there is a coordinate neighborhood  $\bar{\mathcal{U}}$  of  $\xi$  on  $\bar{W}_0$  and a map  $\bar{\varphi} : \bar{\mathcal{U}} \rightarrow U \times V \subset \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$ , and also a neighborhood  $\mathcal{U} = \pi\bar{\mathcal{U}}$  of  $p = \pi\xi$ , together with a map  $\varphi : \mathcal{U} \rightarrow U \subset \mathbb{R}^n$ , and this structure is defined such that  $\varphi \circ \pi = \pi_1 \circ \bar{\varphi}$  where  $\pi_1 : U \times V \rightarrow U$  is the projection on the first component. We also define these maps in such a way that  $\varphi(p) = 0 \in U$  and  $\bar{\varphi}(\xi) = 0 \in U \times V$ .

Let  $\bar{\varphi}(\bar{\gamma}(s)) = (\varphi(s), \beta(s))$ , so that  $\varphi(s) \subset U$  and  $\beta(s) \subset V$  and  $\varphi(\gamma(s)) = \varphi(s)$ . We identify an open neighborhood of the origin in  $T_\xi \bar{W}_0$  with  $U \times V$ , by which we mean that we will write  $\frac{d}{ds}(\bar{f}(\bar{\gamma}(s)))|_{s=s_0} = (\varphi'(s_0), \beta'(s_0))$ . By [3, Theorem 3.2], for vectors  $(X, Y) \in U \times V$  such that  $A_\xi X = \mu X$ , then

$$\bar{A}_t(X, Y) = \frac{\mu}{1 - t\mu}(X, 0) - \frac{1}{t}(0, Y).$$

We chose  $\bar{\gamma}(s)$  in such a way that  $\bar{\gamma}'(s_0) \in \ker \bar{A}_t$ . By the above description of  $\bar{A}_t$ , we see that this forces  $\beta'(s_0) = 0$  and  $\varphi'(s_0) \in \ker A_\xi$ . But  $\varphi'(s_0) = \gamma'(s_0)$ .

Therefore,

$$\frac{d}{ds}(\bar{f}(\bar{\gamma}(s)) - f(\gamma(s)))|_{s=s_0} = 0.$$

Since  $s_0$  was arbitrary,  $\bar{f}(\bar{\gamma}(s)) - f(\gamma(s)) = t\xi$ , where  $\xi$  is a constant vector. Hence, if  $\bar{f}(\bar{\gamma}(s))$  is a straight line segment, then so is  $f(\gamma(s))$ .  $\square$

2.2. *Remark.* If  $M$  is compact, there must be a height function with a non-degenerate critical point on every open neighborhood of  $f(M)$ , since  $f(M)$  cannot contain a straight line segment if it is analytic. Since the above argument is local, the result is true not only for a neighborhood of an analytically immersed compact manifold but also for a neighborhood of every regular point of an analytic subset of  $f(M)$ ; i.e., top-sets of  $f(M)$ .

From now on, let  $f : M^{2k} \rightarrow \mathbb{R}^N$  be a tight analytic substantial immersion of a  $(k - 1)$ -connected manifold, and let  $\Omega \subset f(M)$  be a top-set.

2.3. **Lemma.** *If  $\Omega$  is not a single point, then  $\underline{\dim} \Omega \geq k$ .*

*Proof.* We know that by tightness the map  $\check{H}^*(M) \rightarrow \check{H}^*(f^{-1}(\Omega))$  induced by inclusion must be surjective. Now suppose there is some regular point  $p \in \Omega$  such that  $\dim_p \Omega = m$  and  $m < k$ . Since  $p$  is regular, there is neighborhood of  $p$  in  $\Omega$  which is a submanifold of  $\mathbb{R}^N$ . By Lemma 2.1 we can pick  $p$  such that there is some vector  $\xi \in N_p \Omega$  normal to  $\Omega$  at  $p$  such that  $h_\xi|_\Omega$  has a critical point at  $p$  of positive index  $l$ , with  $0 < l \leq m$  (if the index is zero, replace  $\xi$  by  $-\xi$ ).

Now let  $\mathcal{L} = \{x \in \mathbb{R}^N \mid h_\xi(x - f(p)) \leq 0\}$ . By Lemma 1.1,  $\check{H}^l(f^{-1}(\mathcal{L})) \neq 0$ . Since top\*-sets are tight, the map  $\check{H}^*(f^{-1}(\mathcal{L})) \rightarrow \check{H}^*(f^{-1}(\Omega))$  induced by inclusion is also surjective; hence, the map

$$\check{H}^*(M) \rightarrow \check{H}^*(f^{-1}(\mathcal{L})) \neq 0$$

must be surjective. Since we are assuming that  $M$  is highly connected,  $\check{H}^*(M) \neq 0$  only in dimensions  $*$  = 0,  $k$ ,  $2k$ . Therefore  $\underline{\dim} \Omega \geq k$ .  $\square$

2.4. **Lemma.** *If  $\underline{\dim} \Omega = k$  then  $\Omega$  spans a  $(k + 1)$ -dimensional affine subspace and  $\underline{\dim}_p \Omega = k$  at every  $p \in \Omega$ .*

*Proof.* Suppose that there is a point  $q \in \Omega$  with  $\underline{\dim}_q \Omega = k$ . Then there is some point  $p \in \Omega$  with a  $k$ -dimensional smooth open neighborhood  $K$ , such that there is a height function  $h_\xi$  with  $p$  as a nondegenerate critical point. Then  $p$  must be a nondegenerate convex point, since suppose  $h_\xi$  was a height function with  $p$  as a critical point, but  $p$  was neither a minimum or maximum. Then  $p$  must be of index less than  $k$ , and, since  $M$  is  $(k - 1)$ -connected, this would contradict the fact that  $\Omega$  is tight.

Suppose that  $\Omega$  is substantial in some  $(k + l)$ -dimensional affine space  $E$ . It will be shown that then  $l = 1$ . Suppose not, and that  $l > 1$ . Since  $p$  is a nondegenerate convex point, then  $\alpha(X, X) \neq 0$  for every nonzero  $X \in T_p K$ , and the set  $\mathcal{A} = \{\alpha(X, X) \mid X \in T_p K\}$  is contained in a half space of the  $l$ -dimensional space  $N_p K$ . Denote the convex hull of  $\mathcal{A}$  by  $\mathcal{K}_p$ . It is clear that, since  $\Omega$  is tight,  $\mathcal{K}_p$  spans  $N_p K$ . It follows that extremal supporting hyperplanes to  $\Omega$  at  $p$  are those hyperplanes  $\ell$  in  $E$  containing  $T_p K$ , such that  $h = \ell \cap N_p K$  is an extremal supporting hyperplane of  $\mathcal{K}_p$  in  $N_p K$ .

Now let  $h'$  be a hyperplane in  $N_pK$  which does not support  $\mathcal{H}_p$  and intersects  $\mathcal{H}_p$  in a cone, or, in the case when  $l = 2$ , the intersection is a line in the interior of  $\mathcal{H}_p$ . Then the hyperplane  $\ell'$  in  $E$  spanned by  $T_pK$  and  $h'$  does not support  $\Omega$  at  $p$ . Let  $\xi \in N_pK$  be chosen such that  $\xi \perp \ell'$ . Since  $\xi \perp T_pK$ ,  $p$  is a critical point of  $h_\xi$ , of index  $l$  where  $0 < l < k$ . Then by Remark 1.3, we know that

$$\check{H}^l(f^{-1}(\ell_\xi)) \neq 0, \quad \text{where } \ell_\xi = \{x \in E \mid h_\xi(x - p) \leq 0\}.$$

Since  $M$  and, hence,  $\Omega$  are  $(k - 1)$ -connected, the map induced by inclusion  $\check{H}^l(f^{-1}(\Omega)) \rightarrow \check{H}^l(f^{-1}(\ell_\xi))$  cannot be surjective. Hence it is not possible to select a hyperplane  $h'$  in  $N_pK$  which does not support  $\mathcal{H}_p$ , and so  $\mathcal{H}_p$  must be one-dimensional. Since  $\mathcal{H}_p$  spans  $N_pK$ ,  $\dim N_pK = 1$ ,  $l = 1$ , and  $\Omega$  is contained in a  $(k + 1)$ -dimensional affine space.

Lastly, by Remark 1.7, if  $\Omega$  had a point with a  $(k + 1)$ -dimensional neighborhood, then  $\Omega$  would have to be the entire affine space  $E$ , which would contradict compactness.  $\square$

Much can be said about top-sets of tight immersions of highly connected manifolds even if we do not assume the immersion is analytic, as shown in the following lemma [3, p. 84].

**2.5. Lemma.** *Let  $f : M^{2k} \rightarrow \mathbb{R}^N$  be a topological immersion of a compact manifold. Assume that, for almost all unit vectors  $\xi$ , the set  $M_r(\xi) = \{x \in f(M) \mid h_\xi(x) \leq r\}$  is  $(k - 1)$ -connected for all  $r \in \mathbb{R}$ , and let  $\Omega$  be a top\*-set.*

- (a) *If  $\Omega$  spans a  $j$ -dimensional affine space, with  $j \leq k$ , then  $\Omega$  is a  $j$ -disk.*
- (b) *If  $\Omega$  spans a  $(k + 1)$ -dimensional affine space, then  $\partial \mathcal{H}\Omega \subset \Omega$ .*

**2.6. Proposition.** *Let  $f : M^{2k} \rightarrow \mathbb{R}^{2k+1}$  be a tight, analytic immersion of a  $(k - 1)$ -connected compact manifold. If  $\Omega$  is a top-set such that  $\underline{\dim} \Omega \leq k$ , then  $\Omega$  is either a point, or  $\Omega$  is a convex  $k$ -cycle.*

*Proof.* We know that, if  $\Omega$  contains any points  $p$  such that  $\underline{\dim}_p \Omega < k$ , then  $\Omega$  must be a single point, and, if  $\underline{\dim}_p \Omega = k$  at one point  $p$ , then  $\underline{\dim}_p \Omega = k$  at every point  $p \in \Omega$ .

So  $\underline{\dim} \Omega = k$  and  $\Omega$  spans a  $(k + 1)$ -dimensional affine space which we will call  $E$ , and by Lemma 2.5(b),  $\partial \mathcal{H}\Omega \subset \Omega$ . We also know that no point  $p \in \Omega$  has a neighborhood that is  $(k + 1)$ -dimensional. The only possibility that must then be ruled out is that  $(\mathcal{H}\Omega)^\circ \cap \Omega \neq \emptyset$ , where  $(\mathcal{H}\Omega)^\circ$  represents the interior of  $\mathcal{H}\Omega$ .

Let  $p \in (\mathcal{H}\Omega)^\circ \cap \Omega$ . Furthermore, assume that  $p$  is a regular point with a  $k$ -dimensional open neighborhood. We will show that this violates the two-piece property. Let  $\xi$  be a unit normal vector in  $E$ , which is normal to  $\Omega$  at a point  $p$ , selected such that  $p$  is a nondegenerate critical point of the height function  $h_\xi$ . Assume that  $h_\xi(p) = 0$ . The hyperplane  $h_\xi^{-1}(0)$  is tangent to  $\Omega$  at  $p$ . By Lemma 2.3, we know that  $p$  must be of index 0 or  $k$ . Then some near parallel hyperplane  $h_\xi(\epsilon)$  for  $\epsilon > 0$  small (depending on the choice of  $\pm\xi$ , we may need to choose  $\epsilon < 0$ ), will meet  $\Omega$  near  $p$ , and it will divide  $\Omega$  into at least three parts: since  $p$  is in the interior of  $\mathcal{H}\Omega$ ,  $h_\xi(\epsilon)$  will divide  $\partial \mathcal{H}\Omega$  into two pieces, but it will also separate from these two, the piece containing  $p$ . This violates the two-piece property.  $\square$

**2.7. Proposition.** *Let  $f : M^{2k} \rightarrow \mathbb{R}^{2k+2}$  be a tight, analytic immersion of a  $(k - 1)$ -connected manifold. Then every extremal supporting hyperplane to  $M$  through a nondegenerate critical point intersects  $M$  in a convex  $k$ -cycle.*

*Proof.* We first want to show that, if  $\ell$  is an extremal supporting hyperplane to  $f(M)$  at a nondegenerate critical point  $p$ , then  $\ell \cap f(M)$  must contain more than just the point  $p$ .

Following the notation used previously, since  $p$  is a nondegenerate critical point, the set  $\mathcal{A} = \{\alpha(X, X) \mid X \in T_p M\}$  is a sector lying in an open half space of the normal plane  $N_p M$ , and the extremal supporting hyperplanes at  $p$  are spanned by  $T_p M$  and rays on the boundary of  $\mathcal{A}$ .

Now let  $\ell$  be an extremal supporting hyperplane at  $p$ . Let us assume that the top-set corresponding to  $\ell$  consists of  $p$  only. Let  $(\ell_n)$  be a sequence of hyperplanes containing  $T_p M$  that are not supporting and converge to  $\ell$ . We assume that the sequence  $(\ell_n)$  is strictly monotone on the pencil of hyperplanes containing  $T_p M$ . Let  $(\mathcal{L}_n)$  be the sequence of closed half spaces such that  $\mathcal{L}_n$  has  $\ell_n$  as boundary and such that

$$f^{-1}(\mathcal{L}_1) \supset \cdots \supset f^{-1}(\mathcal{L}_n) \supset \cdots \supset f^{-1}(\ell) = \{p\}.$$

Notice that

$$\{p\} = \bigcap f^{-1}(\mathcal{L}_n).$$

Let  $n$  be so big that an  $\epsilon$ -neighborhood  $B_\epsilon(f^{-1}(\mathcal{L}_n))$  of  $f^{-1}(\mathcal{L}_n)$  does not contain any  $l$ -cycle that is nontrivial in  $M$ . Now let  $v_n \in N_p M$  be orthogonal to  $\ell_n$ . Then  $p$  is a nondegenerate critical point of  $h_{v_n}$  of index  $l$ , and, by Lemma 1.5, in fact  $l = k$ . There is close to  $h_{v_n}$  a height function  $h_\xi$  which is a Morse function such that close to  $p$  is a critical point  $q$  of index  $k$  and  $U \subset B_\epsilon(f^{-1}(\mathcal{L}_n))$ , where  $U = f^{-1}((-\infty, h_\xi(q)])$ . Now  $U$  contains a  $k$ -cycle that is nontrivial in  $M$ . This is a contradiction, and hence  $f^{-1}(\ell)$  cannot consist of one point only. Let  $\Omega = \ell \cap f(M)$ .

Now let  $h = \ell \cap \mathcal{A}$ , and  $\mathcal{E}_h = \{X \in T_p M \mid \alpha(X, X) \in h\}$ . It is an immediate consequence of Lemmas 1.2 and 1.5 that  $\dim \mathcal{E}_h = k$ . We also know that  $\underline{\dim} \Omega \geq k$ . Then, if  $p \in \text{reg } \Omega$ , then  $\underline{\dim} \Omega = k$ ,  $T_p \Omega = \mathcal{E}_h$ , and, by Proposition 2.6, we know that  $\Omega$  is a convex  $k$ -cycle. Even if  $p \in \text{sng } \Omega$ , it is still clear that  $\underline{\dim}_p \Omega = k$ , and, by Proposition 2.6,  $\Omega$  must be a convex  $k$ -cycle, and  $\partial \Omega = \emptyset$ .  $\square$

### 3. PROOF OF THEOREM

*Proof.* We know that in general  $\omega_k(M^{2k}) = 0$  for  $(k - 1)$ -connected manifolds with  $k \neq 1, 2, 4$ , or  $8$ . Thorbergsson [9] showed that for  $k = 4$  or  $8$ , the codimension must be  $k + 1$  or  $k + 2$ . Since we are assuming that the codimension is 2, it is only necessary to make the restriction  $\omega_k(M^{2k}) = 0$  in the case when  $k = 1$  or  $2$ , since it is automatically true in all other cases.

Following the notation used in the proof of Proposition 2.7, since the normal space  $N_p M$  is two dimensional, the set  $\mathcal{A}$  is a sector, which cannot degenerate to a line. The sector is bounded by two vectors,  $h_1$  and  $h_2$ , and these vectors, along with  $T_p M$ , span extremal supporting hyperplanes,  $\ell_1$  and  $\ell_2$ . These hyperplanes intersect  $f(M)$  in strictly convex  $k$ -cycles, call them  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . It was shown in the proof of Proposition 2.7 that these cycles have well-defined

tangent spaces, and these tangent spaces map under  $\alpha$  to the rays  $h_1$  and  $h_2$ . It then follows that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  meet transversally at  $p$  and nowhere else.

Let  $U$  be a neighborhood of  $p$  such that every point in  $U$  is a nondegenerate convex point. Then, there are two families of top cycles through every point of  $U$ . Denote these families by  $\mathcal{F}_U^1$  and  $\mathcal{F}_U^2$ , and for each point  $q \in U$  denote the top cycle of the respective family by  $\mathcal{S}_1(q)$  and  $\mathcal{S}_2(q)$ . It follows from previous arguments that  $\mathcal{S}_1(q)$  and  $\mathcal{S}_2(q)$  depend continuously on  $q$ , and any two cycles from the same family are homologous.

Next we prove that two different cycles in the same family must be disjoint. Suppose that  $\mathcal{S}$  and  $\overline{\mathcal{S}}$  are two different cycles in the same family  $\mathcal{F}_U^i$  such that  $q \in \mathcal{S} \cap \overline{\mathcal{S}}$ . Let  $\ell$  and  $\overline{\ell}$  be the extremal supporting hyperplanes that meet  $f(M)$  in  $\mathcal{S}$  and  $\overline{\mathcal{S}}$ . Then  $\ell$  and  $\overline{\ell}$  intersect in  $T_q M$ , and hence the intersection  $\mathcal{S} \cap \overline{\mathcal{S}}$  is a top-set and, by Proposition 2.7, must be a single point, call it  $p$ . Let  $K(p) = \text{Span}\{\gamma'(0) | \gamma : (-\epsilon, \epsilon) \rightarrow M, \gamma \in C^\infty, \gamma(0) = p, \gamma([0, \epsilon)) \subset \mathcal{S}\} \subset T_p M$ . We can similarly define  $\overline{K}(p)$  with respect to  $\overline{\mathcal{S}}$ . If  $p$  is a regular point, then  $K(p) = T_p \mathcal{S}$ . Since we are assuming that  $\omega_k(M^{2k}) = 0$ , then  $\text{Span}(K(p), \overline{K}(p)) \subsetneq T_p M$ , and so  $B = K(p) \cap \overline{K}(p)$  is some affine subspace with  $k \geq \dim(B) > 0$  and  $B \subset \ell \cap \overline{\ell}$ . Therefore,  $\ell \cap \overline{\ell}$  meets  $M$  in more than just one point, which is a contradiction.

We now know that the families  $\mathcal{F}_U^i$  cover ribbons of the manifold  $M$  simply. These cylinders are homeomorphic to cylinders  $S^k \times A_i$ , where  $A_i = U/\mathcal{F}_U^i$ . Denote these ribbons by  $\mathcal{R}_i$ , and notice that they consist completely of top points.

Denote the set of nondegenerate convex points of  $\mathcal{R}_i$  by  $V_i$ . We now prove that  $V_i$  is dense in  $\mathcal{R}_i$ . Assume that  $\mathcal{R}_i - \overline{V}_i \neq \emptyset$ . Then there is some connected open set  $W_i \subset \mathcal{R}_i$  such that the open sets  $W_i \cap V_i$  and  $W_i - \overline{V}_i$  are nonempty. Suppose that  $(x_1, x_2, \dots, x_{2k})$  is a coordinate system on  $W$ . Then define

$$e_i = \frac{\partial f}{\partial x_i} \quad \text{for } i = 1, 2, \dots, 2k,$$

$$e_{jl} = \frac{\partial^2 f}{\partial x_j \partial x_l} \quad \text{for } j, l = 1, 2, \dots, 2k.$$

At a nondegenerate convex point  $p \in W_i$ , the set of vectors  $\{e_i(p), e_{jl}(p)\}_{i,j,l=1}^{2k}$  span  $\mathbb{R}^{2k+2}$ . This is equivalent to the observation in the proof to Proposition 2.7, that the sector  $\mathcal{A}$  does not degenerate to a ray. Select  $2k+2$  linearly independent vectors from this set and relabel them  $\overline{e}_{i_1}, \dots, \overline{e}_{i_{2k+2}}$ . Now define an analytic function  $\Phi$  on  $W_i$  such that  $\Phi$  is the determinant of  $(\overline{e}_{i_1}, \dots, \overline{e}_{i_{2k+2}})$ . Then  $\Phi$  vanishes identically on the nonempty open set  $W_i - \overline{V}_i$  of top points which are not nondegenerate convex points. But since it is analytic, it must vanish identically, which is a contradiction since  $\Phi(p) \neq 0$ . Hence, the nondegenerate convex points are dense in  $\mathcal{R}_i$ .

Notice that through every nondegenerate convex point in the ribbon  $\mathcal{R}_2$  pass two convex  $k$ -cycles, and one of them belongs to the family  $\mathcal{F}_U^2$ . Suppose that a point  $q \in \mathcal{R}_2$  is not a nondegenerate convex point. Then, since the nondegenerate convex points are dense in  $\mathcal{R}_2$ , we know that there is a sequence of nondegenerate convex points  $q_n \rightarrow q$ . Then a subsequence of the convex

$k$ -cycles  $(\mathcal{S}_1(q_n))$  converges to a  $k$ -cycle  $\mathcal{S}$  that cannot degenerate to anything other than a  $k$ -sphere, since it must carry a nontrivial  $k$ -cycle. The convex cycle  $\mathcal{S}$  lies in a top-set, and as a result of Lemma 2.5 and Proposition 2.7, it is in fact the whole top-set.

This convex cycle does not depend on the particular choice of sequence  $(q_n)$ , since otherwise there would be at least three top cycles through  $q$ , and we have shown that there can only be two. We are now able to extend the family  $\mathcal{F}_U^1$  to a continuous family  $\mathcal{F}^1$  of top cycles that cover  $M$  simply. We do this by defining  $\mathcal{F}^1$  to be all cycles through points in  $\mathcal{B}_2$  which are not also top cycles in the family  $\mathcal{F}_U^2$ .

As shown previously, two cycles in the family  $\mathcal{F}^1$  cannot have a point in common, and clearly  $\mathcal{F}^1$  covers an open and closed subset of  $M$ , and so must be all of  $M$ . The family  $\mathcal{F}_U^2$  can be extended to  $\mathcal{F}^2$  in the same way. It is also clear that every top cycle of  $\mathcal{F}^1$  meets every cycle of  $\mathcal{F}^2$  since their intersection number depends only on their homology class, and they only meet once by the previous argument.

To complete the proof, we define a homeomorphism between  $M$  and  $S^k \times S^k$ . This is accomplished by associating to  $q \in M$  a point in  $\mathcal{S}_1(p) \times \mathcal{S}_2(p)$  given by  $(\mathcal{S}_1(p) \cap \mathcal{S}_2(q), \mathcal{S}_1(q) \cap \mathcal{S}_2(p))$ . This completes the proof since the cycles  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are homeomorphic to  $k$ -spheres.  $\square$

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