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TYCHONOFF'S THEOREM

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ABSTRACT. A simple proof of Tychonoff's Theorem is given. The proof is, in spirit, much like Tychonoff's original proof, which is also given.

Tychonoff's Theorem states that the arbitrary product of compact spaces is compact. There are two standard proofs of this theorem found in topology books. One proof uses Alexander's Lemma, which states that a space is compact if every cover by a subbasis has a finite subcollection that covers [1, p. 139; 2, p. 4]. An alternative proof by Bourbaki [1, p. 143; 3, pp. 232–233] uses a formulation of compactness in terms of closed sets instead of open sets. Of course, each of these proofs will show that the product of two compact spaces is compact. However, in each case the proof is far more complicated than any standard simple proof that the product of two compact spaces is compact. Indeed, Munkres [3, p. 229] seems to think that the Tychonoff Theorem is a "deep" theorem with no straightforward proof.

In this note we give two simple proofs that the product of two compact spaces is compact. Each of these proofs generalizes easily to Tychonoff's Theorem using only the fact that any set can be well ordered. In the special case of the countably infinite product of compact spaces, this is just ordinary mathematical induction.

The first proof is Tychonoff's original proof [4]. Tychonoff's proof uses a nontrivial alternative formulation of compactness that seems almost forgotten among modern day topologists [2, p. 4]. The second proof uses an obvious formulation of compactness that is only slightly different form the usual covering definition. This proof is, in spirit, much like Tychonoff's. A slight variation of the second proof has been known and used by professors and students at the University of Wisconsin for many (over thirty) years. But, to my knowledge, it is almost unknown to others.

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Definitions

For completeness, we provide definitions of products and the product topology. Let $\{X_\alpha | \alpha \in J\}$ be a collection of sets. The product $\prod_{\alpha \in J} X_\alpha$ is defined to be the collection of all functions $f\colon J \to \bigcup_{\alpha \in J} X_\alpha$ such that $f(\alpha) \in X_\alpha$.

We often write $f = (x_\alpha)_{\alpha \in J}$ or simply (x_α) where $f(\alpha) = x_\alpha$. For each α there is a *projection function* onto X_α, $P_\alpha\colon \prod_{\alpha \in J} X_\alpha \to X_\alpha$ defined by $P_\alpha(f) = f(\alpha)$. If each X_α is a topological space, then the *product topology* on $\prod_{\alpha \in J} X_\alpha$ is the *smallest* topology, which makes the projection functions continuous. Let F be a finite subset of J, and for each $\alpha \in F$ let U_α be an open subset of X_α. Sets of the form $U = \bigcap_{\alpha \in F} P^{-1}(U_\alpha)$ form a basis for the product topology.

Formulations of compactness

Definition. A topological space X is said to be compact if every open covering of X has a finite subcollection that covers.

The following are equivalent formulations of compactness:

(A) *A topological space X is compact if and only if for each collection of open sets with the property that no finite subcollection covers there is a point $x \in X$ so that x is not covered by the collection of open sets.*

(B) *A topological space X is compact if and only if for each collection of closed subsets of X with the finite intersection property* (*the intersection of finitely many elements of the collection is nonempty*) *the intersection of all elements of the collection is nonempty.*

Definition. Let E be a subset of a topological space. We say that a limit point x of E is a *perfect limit point* of E if for every neighborhood U of x the cardinality of $U \cap E$ is the same as the cardinality of E.

(C) *A topological space X is compact if and only if each infinite subset E of X has a perfect limit point.*

The proofs of (A) and (B) are immediate. Alexander's Lemma uses (A). The Bourbaki proof uses (B). Our simple proof uses (A). Tychonoff used (C), and this fact requires some elementary cardinal arithmetic, which probably explains why it is not well-known. We give a proof here for completeness.

Proof of (C). Suppose X is compact and E is an infinite set with no perfect limit point. For each point x of X choose a neighborhood U_x so that the cardinality of $U_x \cap E$ is less than the cardinality of E. A finite subcollection $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ covers X. Then E is the finite union of $U_{x_i} \cap E$. But the finite union of sets of cardinality less than E must also have cardinality less than E, which is a contradiction.

Suppose every infinite set has a perfect limit point. If X fails to be compact, there is an infinite collection $\{U_\alpha | \alpha \in J\}$ of open subsets of X which covers X and so that no finite subcollection covers. We may also assume that the set J has the minimum cardinality with this property. We further suppose that J is well ordered so that for each α the cardinality of $\{\beta \in J | \beta < \alpha\}$ is less than the cardinality of J and $U_\alpha \not\subset \bigcup\{U_\beta | \beta < \alpha\}$. We define a set $E = \{x_\alpha | \alpha \in J\}$ so that $x_\alpha \in U_\alpha \setminus \bigcup\{U_\beta | \beta < \alpha\}$. The cardinality of E is the same as the cardinality of J. If x is a point of X, then x lies in some U_α, but the cardinality of $U_\alpha \cap E$ is less than the cardinality of E, contradicting the fact that every infinite set has a perfect limit point.

Tychonoff's proof

Theorem. *Let X and Y be compact spaces then $X \times Y$ is compact.*

Proof. Let E be an infinite subset of $X \times Y$. We first show that there is an $a \in X$ so that for each neighborhood U of a the cardinality of $(U \times Y) \cap E$ is the same as the cardinality of E. If no such a exists, then for each $x \in X$ there exists an open set U_x containing x so that $(U_x \times Y) \cap E$ has cardinality less than E. By compactness a finite subcollection $U_{x_1}, U_{x_2}, \ldots, U_{x_m}$ covers X. Hence, $E = (X \times Y) \cap E = ((U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_m}) \times Y) \cap E = \bigcup_{i=1}^m ((U_{x_i} \times Y) \cap E)$. This is a contradiction since the infinite set E cannot be written as the finite union of sets of cardinality less than E.

Similarly we can show that there is a $b \in Y$ so that, for each basic open set of the form $U \times V$ containing (a, b), $(U \times V) \cap E$ has the same cardinality as E. This implies that (a, b) is a perfect limit point of E, so $X \times Y$ is compact.

The simple proof

Theorem. *Let X and Y be compact spaces then $X \times Y$ is compact.*

Proof. Let $\mathscr{G}$ be a collection of open sets of $X \times Y$ so that no finite subcollection of $\mathscr{G}$ covers. In a manner similar to the above proof, we first show that there is an $a \in X$ so that for each neighborhood U of a no finite subcollection of $\mathscr{G}$ covers $U \times Y$. We then show that there is a $b \in Y$ so that for each basic open set of the form $U \times V$ containing (a, b) no finite subcollection of $\mathscr{G}$ covers $U \times V$. Thus the point (a, b) is not covered by any element of $\mathscr{G}$, and we see that $X \times Y$ is compact.

Tychonoff's Theorem. *Let $\{X_\alpha | \alpha \in J\}$ be a collection of compact topological spaces. Then the product $\prod_{\alpha \in J} X_\alpha$ is compact.*

Each of the preceding proofs generalizes easily. We generalize the second proof.

Proof. We assume that J is well ordered and that a covering $\mathscr{G}$ is given so that no finite subcollection of $\mathscr{G}$ covers. Inductively define $a_\gamma \in X_\gamma$ so that, if U is any basic open set containing $\prod_{a \leq \gamma} \{a_\alpha\} \times \prod_{\alpha > \gamma} X_\alpha$, then no finite subcollection of $\mathscr{G}$ covers U. Thus the point (a_α) is not covered by any element of $\mathscr{G}$. Hence, the product is compact.

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