

## AN INVARIANT RIEMANN TYPE INTEGRAL DEFINED BY FIGURES

ALEXEI NOVIKOV AND WASHEK F. PFEFFER

(Communicated by Andrew M. Bruckner)

**ABSTRACT.** We show that a multidimensional generalized Riemann integral defined by means of rectangular figures is already invariant with respect to lipeomorphic changes of coordinates.

In recent years, following the work of Henstock, Kurzweil, and Mawhin, several authors proposed definitions of a generalized Riemann integral that can be used to establish the Stokes theorem for noncontinuously differentiable forms. All these definitions are based on concepts which are a priori invariant with respect to changes of coordinates, e.g., sets of finite perimeter [P1, P2] or partitions of unity [JK1, JK2, KMP]. In this note we show that an integral defined by means of figures, i.e., finite unions of intervals, is already invariant with respect to lipeomorphisms.

### 1. PRELIMINARIES

Throughout this note,  $m \geq 1$  is a fixed integer, and the metric in  $\mathbf{R}^m$  is induced by the norm  $|x| = \max\{|\xi_1|, \dots, |\xi_m|\}$ . If  $E \subset \mathbf{R}^m$ , then  $d(E)$ ,  $E^-$ ,  $E^\circ$ , and  $\partial E$  denote, respectively, the diameter, closure, interior, and boundary of  $E$ . In  $\mathbf{R}^m$  we consider the Lebesgue measure  $\lambda$  and the  $(m-1)$ -dimensional Hausdorff measure  $\mathcal{H}$ . We say that the sets  $A, B \subset \mathbf{R}^m$  *overlap* whenever  $\lambda(A \cap B) > 0$ . A set  $T \subset \mathbf{R}^m$  is called *thin* if it is of  $\sigma$ -finite measure  $\mathcal{H}$ .

*DeGiorgi's perimeter*, finite or infinite, of a bounded Borel set  $E \subset \mathbf{R}^m$  is denoted by  $\|E\|$  (see [Z, Definition 5.4.1]). By  $\mathcal{K}$  we denote the family of all compact sets  $K \subset \mathbf{R}^m$  with  $\mathcal{H}(\partial K) < +\infty$ . According to [Z, Theorem 5.8.5],  $\|K\| < \mathcal{H}(\partial K)$  for each  $K \in \mathcal{K}$ .

An *interval* is always a compact nondegenerate subinterval of  $\mathbf{R}^m$ , i.e., the product  $C = \prod_{j=1}^m [a_j, b_j]$  where  $a_j < b_j$  are real numbers; when  $b_j - a_j = d$  for  $j = 1, \dots, m$ , we say that  $C$  is a *cube* of diameter  $d(C) = d$ . A *figure* is a finite, possibly empty, union of intervals. Each figure  $A$  belongs to  $\mathcal{K}$ , and  $\|A\| = \mathcal{H}(\partial A)$ . If  $A$  and  $B$  are figures, then so are the sets  $A \cup B$  and  $(A - B)^-$ . The number

$$r(A) = \frac{\lambda(A)}{d(A)\|A\|}$$

---

Received by the editors June 29, 1992.

1991 *Mathematics Subject Classification.* Primary 26A39, 26B15.

is called the *regularity* of a nonempty figure  $A$ .

All functions we consider are real-valued. A nonnegative function  $\delta$  on a set  $E \subset \mathbf{R}^m$  is called a *gage* on  $E$  whenever the set  $\{x \in E : \delta(x) = 0\}$  is thin. A *caliber* is a sequence  $\{\eta_j\}$  of positive real numbers.

A *partition* in a figure  $A$  is a collection  $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ , possibly empty, where  $A_1, \dots, A_p$  are nonoverlapping subfigures of  $A$  and  $x_i \in A_i$  for  $i = 1, \dots, p$ . Given  $\varepsilon > 0$ , a gage  $\delta$  on  $A$ , and a caliber  $\eta = \{\eta_j\}$ , we say that  $P$  is

- (1)  $\varepsilon$ -regular if  $r(A_i) > \varepsilon$  for  $i = 1, \dots, p$ ;
- (2)  $\delta$ -fine if  $d(A_i) < \delta(x_i)$  for  $i = 1, \dots, p$ ;
- (3)  $(\varepsilon, \eta)$ -approximating if  $(A - \bigcup_{i=1}^p A_i)^- = \bigcup_{j=1}^k B_j$  where  $B_1, \dots, B_k$  are nonoverlapping figures with  $\|B_j\| < 1/\varepsilon$  and  $\lambda(B_j) < \eta_j$  for  $j = 1, \dots, k$ .

**Lemma 1.1.** *Let  $\delta$  be a gage in a figure  $A$ . Then for each positive  $\varepsilon < 1/(2m)$  and each caliber  $\eta$  there is a  $\delta$ -fine  $\varepsilon$ -regular and  $(\varepsilon, \eta)$ -approximating partition in  $A$ .*

This existence result has been proved in [P3, §4]. We shall also need a lemma, which follows from [Fa, Theorem 5.1].

**Lemma 1.2.** *There is a constant  $\kappa$ , which depends only on  $m$ , and has the following property: if  $E \subset \mathbf{R}^m$  and  $\mathcal{H}(E) < a$ , then for each  $\eta > 0$  we can find a finite or infinite sequence of cubes  $\{C_n\}$  of diameters less than  $\eta$  and such that  $E \subset \bigcup_n C_n^\circ$  and  $\sum_n \|C_n\| < \kappa a$ .*

## 2. THE INTEGRAL

**Definition 2.1.** A function  $f$  defined on a figure  $A$  is called *integrable* in  $A$  if there is a real number  $I$  having the following property: given  $\varepsilon > 0$ , we can find a gage  $\delta$  on  $A$  and a caliber  $\eta$  so that

$$\left| \sum_{i=1}^p f(x_i) \lambda(A_i) - I \right| < \varepsilon$$

for each  $\delta$ -fine  $\varepsilon$ -regular and  $(\varepsilon, \eta)$ -approximating partition  $\{(A_1, x_1), \dots, (A_p, x_p)\}$  in  $A$ .

It follows from Lemma 1.1 that the number  $I$  of Definition 2.1 is determined uniquely by the integrable function  $f$ . We call it the *integral* of  $f$  over  $A$  and denote it by  $\int_A f$ .

The basic properties of the integral are established in [P3, §§6 and 7]. For the reader's convenience we restate the main results needed for the change of variables theorem (Theorem 3.3).

A function  $F$  defined on a family  $\mathcal{F}$  of bounded Borel sets is called:

- (1) *additive* if  $F(A) = \sum_{D \in \mathcal{D}} F(D)$  for each set  $A \in \mathcal{F}$  and each nonoverlapping family  $\mathcal{D} \subset \mathcal{F}$  with  $\bigcup \mathcal{D} = A$ ;
- (2) *continuous* if, given  $\varepsilon > 0$ , there is an  $\eta > 0$  such that  $|F(A)| < \varepsilon$  for each  $A \in \mathcal{F}$  with  $\|A\| < 1/\varepsilon$  and  $\lambda(A) < \eta$ .

**Proposition 2.2.** *Let  $f$  be an integrable function in a figure  $A$ . Then  $f$  is integrable in each figure  $B \subset A$  and the map  $F : B \mapsto \int_B f$  is an additive continuous function.*

**Proposition 2.3.** *Let  $f$  be a function on a figure  $A$ . Then  $f$  is integrable in  $A$  if and only if there is an additive continuous function  $F$  defined on all subfigures of  $A$  and having the following property: given  $\varepsilon > 0$ , we can find a gage  $\delta$  on  $A$  so that*

$$\sum_{i=1}^p |f(x_i)\lambda(A_i) - F(A_i)| < \varepsilon$$

for each  $\delta$ -fine  $\varepsilon$ -regular partition  $\{(A_1, x_1), \dots, (A_p, x_p)\}$  in  $A$ .

Proposition 2.2 is identical to [P3, Proposition 6.4]. Using the additivity and continuity of  $F$ , it is easy to show that the condition of Proposition 2.3 is sufficient; that it is also necessary follows from Henstock's lemma [P3, Proposition 6.5]. The function  $F$  of Proposition 2.2 coincides with that of Proposition 2.3, and we call it the *indefinite integral* of  $f$  in  $A$ .

### 3. THE INVARIANCE

**Lemma 3.1.** *Let  $K \in \mathcal{K}$  and  $\mathcal{H}(\partial K) < a$ . There is a sequence  $\{A_n\}$  of figures such that  $A_n \subset K^\circ$  and  $\|A_n\| < \kappa a$ , for  $n = 1, 2, \dots$ , and  $\lim \lambda(K - A_n) = 0$ .*

*Proof.* Since  $\partial K$  is compact, it follows from Lemma 1.2 that for each integer  $n \geq 1$  there are cubes  $C_1, \dots, C_k$  of diameters less than  $1/n$  and such that  $\partial K \subset \bigcup_{i=1}^k C_i^\circ$  and  $\sum_{i=1}^k \|C_i\| < \kappa a$ . Then  $A_n = (K - \bigcup_{i=1}^k C_i)^-$  is a subfigure of  $K^\circ$  with  $\|A_n\| < \kappa a$  and

$$\lambda(K - A_n) \leq \sum_{i=1}^k \lambda(C_i) \leq \frac{1}{n} \cdot \frac{1}{2m} \sum_{i=1}^k \|C_i\| < \frac{1}{n} \cdot \frac{\kappa a}{2m}.$$

**Proposition 3.2.** *Let  $F$  be an additive continuous function defined on all subfigures of a figure  $A$ . Then  $F$  has a unique additive continuous extension, still denoted by  $F$ , to the family  $\{K \in \mathcal{K} : K \subset A\}$ .*

*Proof.* On the family  $\mathcal{B}$  of all Borel subsets of  $A$  we consider a metric

$$\rho(B, C) = \lambda[(B - C) \cup (C - B)]$$

and denote by  $\text{cl } \mathcal{E}$  the  $\rho$ -closure of a family  $\mathcal{E} \subset \mathcal{B}$ . According to [Z, Corollary 5.3.4], for  $n = 1, 2, \dots$ , the metric  $\rho$  is complete on the family  $\mathcal{B}_n = \{B \in \mathcal{B} : \|B\| \leq n\}$ . Since

$$\begin{aligned} |F(B) - F(C)| &= |F[(B - C)^-] - F[(C - B)^-]| \\ &\leq |F[(B - C)^-]| + |F[(C - B)^-]| \end{aligned}$$

for all figures  $B, C \subset A$ , the function  $F$  is uniformly  $\rho$ -continuous on the family  $\mathcal{F}_n$  of all figures contained in  $\mathcal{B}_n$ . Consequently,  $F$  has a unique  $\rho$ -continuous extension to  $\text{cl } \mathcal{F}_n$ . As  $\{\mathcal{F}_n\}$  is an increasing sequence,  $F$  has an additional unique extension to  $\mathcal{F} = \bigcup_{n=1}^{\infty} \text{cl } \mathcal{F}_n$ , still denoted by  $F$ . It is easy to verify that  $F$  is additive and continuous on  $\mathcal{F}$ . As  $\mathcal{K} \subset \mathcal{F}$  by Lemma 3.1, the proposition is proved.

Let  $E \subset \mathbb{R}^m$  be a measurable set. For a Lipschitz map  $\Phi: E \rightarrow \mathbb{R}^m$ , we denote by  $\det \Phi$  the determinant of the differential  $D\Phi$  of  $\Phi$ . By the Kirszbraun and Rademacher theorems [Fe, Theorems 2.10.43 and 3.1.6] the function  $\det \Phi$  is defined almost everywhere in  $E$ , and by [P2, Lemma 5.16] it is determined uniquely by  $\Phi$  up to a set of measure zero. A Lipschitz map  $\Phi: E \rightarrow \mathbb{R}^m$  is called a *lipeomorphism* if it is injective and the inverse map  $\Phi^{-1}: \Phi(E) \rightarrow \mathbb{R}^m$  is also Lipschitz.

**Theorem 3.3.** *Let  $\Phi: A \rightarrow B$  be a lipeomorphism from a figure  $A$  onto a figure  $B$ , and let  $f$  be an integrable function on  $B$ . Then  $f \circ \Phi \cdot |\det \Phi|$  is integrable on  $A$  and*

$$\int_A f \circ \Phi \cdot |\det \Phi| = \int_B f.$$

*Proof.* There are positive constants  $\alpha$  and  $\beta$  such that

$$\alpha|x - x^*| \leq |\Phi(x) - \Phi(x^*)| \leq \beta|x - x^*|$$

for all  $x, x^* \in A$ . If  $C$  is a subfigure of  $A$ , then it follows from [Fa, Lemma 1.8] that  $\Phi(C)$  belongs to  $\mathcal{H}$  and satisfies the inequalities

$$\lambda[\Phi(C)] \geq \alpha^m \lambda(C) \quad \text{and} \quad \|\Phi(C)\| \leq \mathcal{H}[\partial\Phi(C)] \leq \beta^{m-1} \|C\|.$$

By Propositions 2.2 and 3.2, the indefinite integral of  $f$  in  $B$  has a unique additive and continuous extension  $F$  to the family  $\{K \in \mathcal{H} : K \subset B\}$ . In view of the above inequalities, the map  $G: C \mapsto F[\Phi(C)]$  is an additive continuous function defined on all subfigures of  $A$ . We use Proposition 2.3 to show that  $G$  is an indefinite integral of  $f \circ \Phi \cdot |\det \Phi|$  in  $A$ .

To this end, choose an  $\varepsilon > 0$ , and use [R, Theorem 7.2.4] to find a  $\lambda$ -zero set  $N \subset A$  and a positive function  $\Delta$  on  $A$  so that

$$|f \circ \Phi(x)| \cdot |\det \Phi(x) \lambda(C) - \lambda[\Phi(C)]| < \varepsilon \lambda(C)$$

for each  $x \in A - N$  and each figure  $C \subset A$  with  $x \in C$ ,  $r(C) > \varepsilon$ , and  $d(C) < \Delta(x)$ . Since  $\lambda[\Phi(N)] = 0$ , by [P3, Corollary 6.8], we may assume that  $f(y) = 0$  for each  $y \in \Phi(N)$  and, consequently, that the previous inequality holds for all  $x \in A$ .

By Proposition 2.3, there is a gage  $\delta_B$  on  $B$  such that

$$\sum_{i=1}^p |f(y_i) \lambda(B_i) - F(B_i)| < \varepsilon$$

for each  $\delta_B$ -fine  $\varepsilon$ -regular partition  $\{(B_1, y_1), \dots, (B_p, y_p)\}$  in  $B$ . With no loss of generality, we may assume that  $\delta_B(x) = 0$  for each  $x \in \partial B$ . Let  $\varepsilon^* = (\beta/\alpha)^m \kappa \varepsilon$  and define a gage  $\delta_A$  on  $A$  by setting  $\delta_A = \min\{\delta_B \circ \Phi/\beta, \Delta\}$ . Choose a  $\delta_A$ -fine  $\varepsilon^*$ -regular partition  $\{(A_1, x_1), \dots, (A_p, x_p)\}$  in  $A$ . For  $i = 1, \dots, p$ , let  $K_i = \Phi(A_i)$  and  $y_i = \Phi(x_i)$ , and observe that  $d(K_i) < \delta_B(y_i)$  and

$$\frac{\lambda(K_i)}{d(K_i) \mathcal{H}(\partial K_i)} \geq \left(\frac{\alpha}{\beta}\right)^m r(A_i) > \kappa \varepsilon.$$

It follows from Lemma 3.1 and Proposition 3.2 that each  $K_i$  contains a figure  $B_i$  such that

$$r(B_i) > \varepsilon, \quad |f(y_i)| \cdot |\lambda(K_i) - \lambda(B_i)| < \varepsilon/p, \quad |G(A_i) - F(B_i)| < \varepsilon/p.$$

As the figure  $B_i$  may not contain the point  $y_i$ , an additional adjustment is necessary. Fix an integer  $i$  with  $1 \leq i \leq p$ , and observe that  $y_i \in B^\circ$  by the choice of  $\delta_B$ . Thus we can select nonoverlapping cubes  $C_1, \dots, C_{2^m}$  contained in  $B$  whose common vertex is the point  $y_i$ . If  $\{j_1, \dots, j_k\}$  is the set of all indexes  $j$  for which  $x_i \in A_j$ , then  $1 \leq k \leq 2^m$  and we let

$$B_{j_s}^* = \left[ (B_{j_s} \cup C_s) - \bigcup_{r \neq s} C_r \right]^-$$

for  $s = 1, \dots, k$ . Since  $y_i \in K_i$ , the cubes  $C_1, \dots, C_{2^m}$  can be chosen so small that  $d(B_i^*) < \delta_B(y_i)$  and the above inequalities hold when  $B_i$  is replaced by  $B_i^*$ . In particular,  $\{(B_1^*, y_1), \dots, (B_p^*, y_p)\}$  is a  $\delta_B$ -fine  $\varepsilon$ -regular partition in  $B$ , and we obtain

$$\begin{aligned} & \sum_{i=1}^p |f \circ \Phi(x_i) \cdot |\det \Phi(x_i)| \lambda(A_i) - G(A_i)| \\ & \leq \sum_{i=1}^p |f \circ \Phi(x_i)| \cdot ||\det \Phi(x_i)| \lambda(A_i) - \lambda(K_i)| + \sum_{i=1}^p |f(y_i)| \cdot |\lambda(K_i) - \lambda(B_i^*)| \\ & \quad + \sum_{i=1}^p |f(y_i) \lambda(B_i^*) - F(B_i^*)| + \sum_{i=1}^p |F(B_i^*) - G(A_i)| \\ & < \sum_{i=1}^p \varepsilon \lambda(A_i) + p \frac{\varepsilon}{p} + \varepsilon + p \frac{\varepsilon}{p} \leq \varepsilon [\lambda(A) + 3]. \end{aligned}$$

This proves the theorem.

## REFERENCES

- [Fa] K. J. Falconer, *The geometry of fractal sets*, Cambridge Univ. Press, Cambridge, 1985.
- [Fe] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.
- [JK1] J. Jarník and J. Kurzweil, *A nonabsolutely convergent integral which admits transformation and can be used for integration on manifolds*, Czechoslovak Math. J. **35** (1986), 116–139.
- [JK2] ———, *A new and more powerful concept of PU-integral*, Czechoslovak Math. J. **38** (1988), 8–48.
- [KMP] J. Kurzweil, J. Mawhin, and W. F. Pfeffer, *An integral defined by approximating BV partitions of unity*, Czechoslovak Math. J. **41** (1991), 695–712.
- [P1] W. F. Pfeffer, *The multidimensional fundamental theorem of calculus*, J. Austral. Math. Soc. Ser. A **43** (1987), 143–170.
- [P2] ———, *The Gauss-Green theorem*, Adv. in Math. **87** (1991), 93–147.
- [P3] ———, *Lectures on geometric integration and the divergence theorem*, Rend. Istit. Mat. Univ. Trieste (in press).
- [R] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1987.
- [Z] W. P. Ziemer, *Weakly differentiable functions*, Springer-Verlag, New York, 1989.

DEPARTMENT OF MATHEMATICS, MOSCOW STATE UNIVERSITY, MOSCOW, REPUBLIC OF RUSSIA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616

E-mail address: wfpfeffer@ucdavis.edu

E-mail address: wfpfeffer@ucdavis.bitnet