ON THE EXISTENCE OF POSITIVE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

L. H. ERBE AND HAIYAN WANG

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Abstract. We study the existence of positive solutions of the equation $u'' + a(t)f(u) = 0$ with linear boundary conditions. We show the existence of at least one positive solution if $f$ is either superlinear or sublinear by a simple application of a Fixed Point Theorem in cones.

1. Introduction

In this paper we shall consider the second-order boundary value problem (BVP)

$$u'' + a(t)f(u) = 0, \quad 0 < t < 1,$$
$$a u(0) - b u'(0) = 0,$$
$$y u(1) + \delta u'(1) = 0.$$

The following conditions will be assumed throughout:

(A.1) $f \in C([0, \infty), [0, \infty))$,
(A.2) $a \in C([0, 1], [0, \infty))$ and $a(t) \neq 0$ on any subinterval of $[0, 1]$.
(A.3) $\alpha, \beta, \gamma, \delta \geq 0$ and $\rho := \gamma \beta + \alpha \gamma + \alpha \delta > 0$.

The BVP (1.1), (1.2) arises in many different areas of applied mathematics and physics; see [1-3, 6, 12, 13] for some references along this line. Additional existence results may be found in [4, 7, 8, 10, 11]. Our purpose here is to give an existence result for positive solutions to the BVP (1.1), (1.2), assuming that $f$ is either superlinear or sublinear. We do not require any monotonicity assumptions on $f$. To be precise, we introduce the notation

$$f_0 := \lim_{u \to 0} \frac{f(u)}{u}, \quad f_\infty := \lim_{u \to \infty} \frac{f(u)}{u}.$$  

Thus, $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. By a positive solution of (1.1), (1.2) we understand a solution $u(t)$ which is positive on $0 < t < 1$ and satisfies

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the differential equation (1.1) for $0 < t < 1$ and the boundary conditions (1.2). By a change of variable, the existence of a positive solution of (1.1), (1.2) may be shown to be equivalent to the existence of a positive radial solution of the semilinear elliptic equation $\Delta u + g(|x|)f(u) = 0$ in the annulus $R_1 < |x| < R_2$ subject to certain boundary conditions for $|x| = R_1$ and $|x| = R_2$. (Here $|x|$ denotes the Euclidean norm.) We refer to [11] for some additional details.

2. Existence results

The main result of this paper is

**Theorem 1.** Assume (A.1)-(A.3) hold. Then the BVP (1.1), (1.2) has at least one positive solution in the case

(i) $f_0 = 0$ and $f_\infty = \infty$ (superlinear), or
(ii) $f_0 = \infty$ and $f_\infty = 0$ (sublinear).

It will be seen in the proof that Theorem 1 is also valid for the more general equation

$$(1.1)^* \quad u'' + f(t, u) = 0$$

with the same boundary conditions (1.2), provided we assume a certain uniformity with respect to the $t$ variable. We state this more general result as

**Corollary 1.** Assume $f$ is continuous, $f(t, u) \geq 0$ for $t \in [0, 1]$, and $u \geq 0$ with $f(t, u) \not= 0$ on any subinterval of $[0, 1]$ for $u > 0$; and let condition (A.3) hold. Then the BVP $(1.1)^*$, (1.2) has at least one positive solution in the case

(i)* $\lim_{u \to 0^+} \max_{t \in [0, 1]} \frac{f(t, u)}{u} = 0$ and $\lim_{u \to \infty} \min_{t \in [0, 1]} \frac{f(t, u)}{u} = \infty$, or
(ii)* $\lim_{u \to 0^+} \min_{t \in [0, 1]} \frac{f(t, u)}{u} = \infty$ and $\lim_{u \to \infty} \max_{t \in [0, 1]} \frac{f(t, u)}{u} = 0$.

The proof of Theorem 1 will be based on an application of the following Fixed Point Theorem due to Krasnoselskii [9]. The proof of Corollary 1 follows from the proof of Theorem 1 with obvious slight modifications which we shall omit.

**Theorem 2** [4, 9]. Let $E$ be a Banach space, and let $K \subset E$ be a cone in $E$. Assume $\Omega_1, \Omega_2$ are open subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let

$$A: K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial \Omega_1$, and $\|Au\| \geq \|u\|$, $u \in K \cap \partial \Omega_2$ ; or
(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial \Omega_1$, and $\|Au\| \leq \|u\|$, $u \in K \cap \partial \Omega_2$.

Then $A$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We will apply the first and second parts of the above Fixed Point Theorem to the superlinear and sublinear cases, respectively.

**Proof of Theorem 1.** Superlinear case. Suppose then that $f_0 = 0$ and $f_\infty = \infty$. We wish to show the existence of a positive solution of (1.1), (1.2). Now (1.1), (1.2) has a solution $u = u(t)$ if and only if $u$ solves the operator equation

$$u(t) = \int_0^1 k(t, s) a(s) f(u(s)) \, ds := Au(t), \quad u \in C[0, 1].$$
Here $k(t, s)$ denotes the Green's function for the BVP

\begin{align*}
(2.1) \quad u'' &= 0; \\
(2.2) \quad \alpha u(0) - \beta u'(0) &= 0, \\
&\gamma u(1) + \delta u'(1) = 0
\end{align*}

and is explicitly given by

\begin{align*}
k(t, s) &= \begin{cases} \\
\frac{1}{\rho} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1, \\
\frac{1}{\rho} (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t < s \leq 1.
\end{cases}
\end{align*}

We let $K$ be the cone in $C[0, 1]$ given by

\begin{align*}
(2.3) \quad K = \left\{ u \in C[0, 1] : u(t) \geq 0, \quad \min_{1/4 \leq t \leq 3/4} u(t) \geq M\|u\| \right\}
\end{align*}

where $\|u\| = \sup_{[0, 1]} |u(t)|$ and

\begin{align*}
(2.4) \quad M &= \min \left\{ \frac{\gamma + 4\delta}{4(\gamma + \delta)}, \quad \frac{\alpha + 4\beta}{4(\alpha + \beta)} \right\}.
\end{align*}

We define

\begin{align*}
(2.5) \quad \varphi(t) := (\gamma + \delta - \gamma t), \quad \psi(t) := \beta + \alpha t, \quad 0 \leq t \leq 1,
\end{align*}

so that

\begin{align*}
(2.6) \quad k(t, s) &= \begin{cases} \\
\frac{1}{\rho} \varphi(t) \psi(s), & 0 \leq s \leq t \leq 1, \\
\frac{1}{\rho} \varphi(s) \psi(t), & 0 \leq t < s \leq 1.
\end{cases}
\end{align*}

Observe that $k(t, s) \leq \frac{1}{\rho} \varphi(s) \psi(s) = k(s, s), \quad 0 \leq t, s \leq 1$, so that, if $u \in K$, then

\begin{align*}
(2.7) \quad Au(t) &= \int_0^1 k(t, s)a(s)f(u(s)) \, ds \leq \int_0^1 k(s, s)a(s)f(u(s)) \, ds
\end{align*}

and hence

\begin{align*}
(2.8) \quad \|Au\| \leq \int_0^1 k(s, s)a(s)f(u(s)) \, ds.
\end{align*}

Furthermore, for $\frac{1}{4} \leq t \leq \frac{3}{4}$

\begin{align*}
\frac{k(t, s)}{k(s, s)} &= \begin{cases} \\
\frac{\varphi(t)}{\varphi(s)}, & s \leq t, \\
\frac{\psi(t)}{\psi(s)}, & t \leq s;
\end{cases} \geq \begin{cases} \\
\frac{\gamma + 4\delta}{4(\gamma + \delta)}, & s \leq t, \\
\frac{\alpha + 4\beta}{4(\alpha + \beta)}, & t \leq s,
\end{cases}
\end{align*}

so

\begin{align*}
\frac{k(t, s)}{k(s, s)} \geq M, \quad \frac{1}{4} \leq t \leq \frac{3}{4}.
\end{align*}

Hence, if $u \in K$,

\begin{align*}
\min_{1/4 \leq t \leq 3/4} Au(t) &= \min_{1/4 \leq t \leq 3/4} \int_0^1 k(t, s)a(s)f(u(s)) \, ds \\
&\geq M \int_0^1 k(s, s)a(s)f(u(s)) \, ds \geq M\|Au\|.
\end{align*}
Therefore, $AK \subset K$. Moreover, it is easy to see that $A : K \to K$ is completely continuous.

Now, since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leq \eta u$, for $0 < u \leq H_1$, where $\eta > 0$ satisfies

\begin{equation}
\eta \int_0^1 k(s, s)a(s) \, ds \leq 1.
\end{equation}

Thus, if $u \in K$ and $\|u\| = H_1$, then from (2.7) and (2.9)

\begin{equation}
Au(t) \leq \int_0^1 k(s, s)a(s)f(u(s)) \leq \|u\|, \quad 0 \leq t \leq 1.
\end{equation}

Now if we let

\begin{equation}
\Omega_1 := \{u \in E : \|u\| < H_1\}
\end{equation}

then (2.10) shows that

\begin{equation}
\|Au\| \leq \|u\|, \quad u \in K \cap \partial \Omega_1.
\end{equation}

Further, since $f_{\infty} = \infty$, there exists $\tilde{H}_2 > 0$ such that $f(u) \geq \mu u$, $u \geq \tilde{H}_2$, where $\mu > 0$ is chosen so that

\begin{equation}
\mu \int_{1/4}^{3/4} k(1/2, s)a(s) \, ds \geq 1.
\end{equation}

Let $H_2 := \max\{2H_1, \tilde{H}_2/M\}$ and $\Omega_2 := \{u \in E : \|u\| < H_2\}$. Then $u \in K$ and $\|u\| = H_2$ implies

\begin{equation}
\min_{1/4 \leq t \leq 3/4} u(t) \geq M\|u\| \geq \tilde{H}_2
\end{equation}

and so

\begin{align*}
Au(1/2) &= \int_0^1 k(1/2, s)a(s)f(u(s)) \, ds \geq \int_{1/4}^{3/4} k(1/2, s)a(s)f(u(s)) \, ds \\
&\geq \mu \int_{1/4}^{3/4} k(1/2, s)a(s)u(s) \, ds \geq \mu M\|u\| \int_{1/4}^{3/4} k(1/2, s)a(s) \, ds \geq \|u\|.
\end{align*}

Hence, $\|Au\| \geq \|u\|$ for $u \in K \cap \partial \Omega_2$.

Therefore, by the first part of the Fixed Point Theorem, it follows that $A$ has a fixed point in $K \cap \overline{\Omega}_2 \backslash \overline{\Omega}_1$ such that $H_1 \leq \|u\| \leq H_2$. Further, since $k(t, s) > 0$, it follows that $u(t) > 0$ for $0 < t < 1$. This completes the superlinear part of the theorem.

\textbf{Sublinear case.} Suppose next that $f_0 = \infty$ and $f_{\infty} = 0$. We first choose $H_1 > 0$ such that $f(u) \geq \hat{\eta}u$ for $0 < u \leq H_1$, where

\begin{equation}
\hat{\eta}M \int_{1/4}^{3/4} k(1/2, s)a(s) \, ds \geq 1
\end{equation}

($M$ is as in the first part of the proof). Then for $u \in K$ and $\|u\| = H_1$ we have

\begin{align*}
Au(1/2) &= \int_0^1 k(1/2, s)a(s)f(u(s)) \, ds \\
&\geq \int_{1/4}^{3/4} k(1/2, s)a(s)f(u(s)) \, ds \geq \hat{\eta} \int_{1/4}^{3/4} k(1/2, s)a(s)u(s) \, ds \\
&\geq \hat{\eta}M\|u\| \int_{1/4}^{3/4} k(1/2, s)a(s) \, ds \geq \|u\| \quad \text{[by (2.14)].}
\end{align*}
Thus, we may let $\Omega_1 := \{u \in E : \|u\| < H_1\}$ so that

$$\|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial \Omega_1.$$

Now, since $f_\infty = 0$, there exists $\tilde{H}_2 > 0$ so that $f(u) \leq \lambda u$ for $u \geq \tilde{H}_2$ where $\lambda > 0$ satisfies

$$\lambda \int_0^1 k(s, s)a(s) \, ds \leq 1. \quad (2.15)$$

We consider two cases:

Case (i). Suppose $f$ is bounded, say $f(u) \leq N$ for all $u \in (0, \infty)$. In this case choose $H_2 := \max\{2H_1, N \int_0^1 k(s, s)a(s) \, ds\}$ so that for $u \in K$ with $\|u\| = H_2$ we have

$$Au(t) = \int_0^1 k(t, s)a(s)f(u(s)) \, ds \leq N \int_0^1 k(s, s)a(s) \, ds \leq H_2$$

and therefore $\|Au\| \leq \|u\|$.

Case (ii). If $f$ is unbounded, then let $H_2 > \max\{2H_1, \tilde{H}_2\}$ and such that $f(u) \leq f(H_2)$ for $0 < u < H_2$.

(We are able to do this since $f$ is unbounded.)

Then for $u \in K$ and $\|u\| = H_2$ we have

$$Au(t) = \int_0^1 k(t, s)a(s)f(u(s)) \, ds \leq \int_0^1 k(s, s)a(s)f(u(s)) \, ds$$

$$\leq \int_0^1 k(s, s)a(s)f(H_2) \, ds \leq \lambda H_2 \int_0^1 k(s, s)a(s) \, ds \leq H_2 = \|u\|.$$

Therefore, in either case we may put

$$\Omega_2 := \{u \in E : \|u\| < H_2\},$$

and for $u \in K \cap \partial \Omega_2$ we have $\|Au\| \leq \|u\|$. By the second part of the Fixed Point Theorem it follows that BVP (1.1), (1.2) has a positive solution, and this completes the proof of the theorem.

**References**


