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COUNTABLE METACOMPACTNESS IN Ψ-SPACES

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ABSTRACT. We prove under a variety of assumptions including $\mathfrak{c} = \aleph_2$ that, for every maximal almost disjoint family $\mathscr{A}$ of countable subsets of ω_1, $\Psi(\mathscr{A})$ is not countably metacompact. In addition, a first countable, countably metacompact, regular space with a closed discrete set which is not a G_δ is constructed from the mutually consistent assumptions that $\mathfrak{b} = \omega_1$ and there can exist a Q-set.

1. INTRODUCTION

Recall that a space is perfect if each closed subset is a G_δ. The easy but important result that all perfect spaces are countably metacompact raises the natural question: how perfect are countably metacompact spaces? In [Bu2] Burke proved that under PMEA closed discrete sets are G_δ's in first countable countably metacompact T_1 spaces.

Given a maximal almost disjoint (mad) family $\mathscr{A} \subseteq [\omega_1]^\omega$ we define the space $\Psi(\mathscr{A})$ as: $\omega_1 \cup \mathscr{A}$ is the underlying set. Every point in ω_1 is isolated while a typical neighborhood of an $a \in \mathscr{A}$ looks like $\{a\} \cup a \setminus y$ where y is a finite subset of a. Then $\Psi(\mathscr{A})$ is a regular, first countable space and $\mathscr{A}$ is a closed discrete set which is not a G_δ. So if there exists a mad $\mathscr{A}$ such that $\Psi(\mathscr{A})$ is countably metacompact, there would be a nice counterexample to the PMEA result. In [Bu1] Burke raised this question and answered it negatively under the assumption $\mathfrak{a} = \mathfrak{c}$. In this note we again answer the question in the negative under a number of different assumptions, including $\mathfrak{c} = \aleph_2$.

There are only two consistent counterexamples to Burke's PMEA theorem in the literature. In [Sh] Shelah forced a normal countably metacompact ladder system space with a closed discrete set which is not a G_δ, while in [BBu] Balogh and Burke constructed a regular counterexample in a ccc forcing extension. Assuming $\mathfrak{b} = \omega_1$ and there can exist a Q-set, we construct a regular first countable countably metacompact space with a closed discrete set which is not a G_δ. This is the only known regular counterexample to the PMEA result which is not a forcing construction.

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Our terminology and notation are fairly standard. $[\omega_1]^\omega =$ the collection of countably infinite subsets of ω_1. $\mathscr{A}$ indicates an infinite mad family on $[\omega_1]^\omega$. $\mathfrak{a}$ is defined as the minimal cardinality of an infinite mad family on ω. For $f, g \in {}^\omega\omega$, $f \leq^* g$ means $g(n) > f(n)$ for at most finitely many n. $\mathfrak{b}$ is the minimal cardinality of an unbounded family in $({}^\omega\omega, \leq^*)$, and $\mathfrak{d}$ is the minimal cardinality of a dominating family in $({}^\omega\omega, \leq^*)$. $X \subseteq^* Y$ means that $X \backslash Y$ is finite. Given $Y \subseteq \omega_1$, $\mathscr{A} \restriction Y = \{a \cap Y : a \in \mathscr{A} \text{ s.t. } |a \cap Y| = \aleph_0\}$ and $\Psi(\mathscr{A} \restriction Y)$ is the subspace of $\Psi(\mathscr{A})$ determined by $Y \cup (\mathscr{A} \restriction Y)$. For more on $\mathfrak{b}, \mathfrak{d}, \mathfrak{a}$, and $\Psi(\mathscr{A})$ see [vD].

2. Preliminaries

We will use the following formulation of countable metacompactness.

Definition 2.1. A space X is countably metacompact iff for every decreasing sequence $\{D_n : n < \omega\}$ of closed subsets of X such that $\bigcap_{n<\omega} D_n = \varnothing$ there exist open $U_n \supseteq D_n$ such that $\bigcap_{n<\omega} U_n = \varnothing$.

The following lemma follows directly from the definitions.

Lemma 2.2. *Given a mad $\mathscr{A} \subseteq [\omega_1]^\omega$, $\Psi(\mathscr{A})$ is countably metacompact iff for every partition $\{\mathscr{A}_n : n \in \omega\}$ of $\mathscr{A}$ there are $X_n \subseteq \omega_1$ such that $\forall n\, \forall m \geq n\, \forall a \in \mathscr{A}_m$ $(a \subseteq^* X_n$ and $\bigcap_{n<\omega} X_n = \varnothing)$.*

Theorem 2.3. *Suppose $\mathfrak{c}$ is regular. Given a mad $\mathscr{A} \subseteq [\omega_1]^\omega$, if $\exists Z \in [\omega_1]^{\omega_1}$ such that $|\mathscr{A} \restriction Y| = \mathfrak{c}$ for each $Y \in [Z]^{\omega_1}$, then $\Psi(\mathscr{A})$ is not countably metacompact.*

Proof. Enumerate $\{x \in [\omega_1]^\omega : |\mathscr{A} \restriction x| = \mathfrak{c}\}$ as $\{x_\alpha : \alpha \leq \mathfrak{c}\}$. Notice that for each $X \in [\omega_1]^{\omega_1}$ if $|\mathscr{A} \restriction X| = \mathfrak{c}$ then there is an $x \in [X]^\omega$ s.t. $|\mathscr{A} \restriction x| = \mathfrak{c}$. For each $n \in \omega$ we construct $\mathscr{A}_\alpha^n \subseteq \mathscr{A}$ inductively on $\alpha \in \mathfrak{c}$ as follows. Fix $a_n \in \mathscr{A}$ such that $\forall n \neq m$ $a_n \neq a_m$, and $\forall n$ $|a_n \cap x_o| = \aleph_0$. Let $\mathscr{A}_0^n = \{a_n\}$.

Having defined $\mathscr{A}_\beta^n$ for all $\beta \leq \alpha$ s.t.

(i) $|\mathscr{A}_\beta^n| = |\beta|$,

(ii) $\forall n \neq m\, \mathscr{A}_\beta^n \cap \mathscr{A}_\beta^m = \varnothing$,

(iii) $\forall \beta \in \alpha\, \exists a \in \mathscr{A}_\beta^n\, |a \cap x_\beta| = \aleph_0$.

Let $\mathscr{A}' = \bigcup_{\beta\in\alpha}\{\mathscr{A}_\beta^n : n \in \omega,\ \beta \in \alpha\}$. For each $n \in \omega$ choose $a_\alpha^n \in \mathscr{A} - \mathscr{A}'$ distinct such that $|a_\alpha^n \cap x_\alpha| = \aleph_0$. Let $\mathscr{A}_\alpha^n = \bigcup_{\beta\in\alpha}\mathscr{A}_\beta^n \cup \{a_\alpha^n\}$. Finally let $\mathscr{A}_n = \bigcup_{\alpha\in\mathfrak{c}}\mathscr{A}_\alpha^n$. Notice that if $x \in [\omega_1]^\omega$ is such that $|\mathscr{A} \restriction x| = \mathfrak{c}$ then for each n there is an $a \in \mathscr{A}_n$ such that $|a \cap x| = \aleph_0$. Fix $n \in \omega$. If X is such that $\forall m \geq n \forall a \in \mathscr{A}_m$ $a \subseteq^* X$, then $|Z - X|$ is countable. Hence, the $\mathscr{A}_n$ witness that $\Psi(\mathscr{A})$ is not countably metacompact.

The following is a corollary to the proof of Theorem 2.3.

Corollary 2.4 (Burke). $(\mathfrak{a} = \mathfrak{c}) \rightarrow \Psi(\mathscr{A})$ *is not countably metacompact for every mad $\mathscr{A} \subseteq [\omega_1]^\omega$.*

3. Unbounded families and partitions of mad families

By Lemma 2.2, to prove that $\Psi(\mathscr{A})$ is not countably metacompact for some mad $\mathscr{A}$ we must exhibit a nasty (i.e., witnessing not countable metacompactness) partition of $\mathscr{A}$ into countably many pieces. We do this by indexing $\mathscr{A}$

with a family $\mathscr{F} \subseteq {}^\omega\omega$ and proving that if $\mathscr{F}$ has certain nice properties then we can build a nasty partition for $\mathscr{A}$. A similar technique was used by Simon in [S] to build a Frechet space whose square is not Frechet.

Definition 3.1. Given $\mathscr{F} \subseteq {}^\omega\omega$, we say $\mathscr{F}$ is fully unbounded if $\forall S \in [\mathscr{F}]^{|\mathscr{F}|}$ S is unbounded under $\leq^*$.

Recall that if $f, g \in {}^\omega\omega$, then $f \leq^* g \Leftrightarrow \{n: g(n) > f(n)\}$ is finite. Clearly no family of size κ where $\kappa < \mathfrak{b}$ or κ is regular and $> \mathfrak{d}$ can be fully unbounded. However, if $\mathfrak{b} \leq \kappa \leq \mathfrak{d}$ then we have positive, consistent, and independent results.

Theorem 3.2. (i) *There are fully unbounded families of size* $\mathfrak{d}$ *and* $\mathfrak{b}$.

(ii) *Let* $\kappa < \delta < \lambda$ *be regular uncountable cardinals. Then* Con(ZFC) $\rightarrow$ Con(ZFC$+(\mathfrak{b} = \kappa) + (\mathfrak{d} = \lambda)$ *and there is no fully unbounded family of size* δ).

(iii) *Let* κ *be a regular uncountable cardinal. Then* Con(ZFC) $\rightarrow$ Con(ZFC $+ (\mathfrak{b} = \aleph_1) + (\mathfrak{d} = \mathfrak{c} = \kappa)$ *and there is a fully unbounded family of size* δ *for each uncountable* $\delta < \mathfrak{c}$).

Proof. (i) Fix a well-ordered unbounded family of type $\mathfrak{b}$ and a dominating family $\{f_\alpha: \alpha < \mathfrak{d}\}$ such that $\alpha < \beta \rightarrow f_\beta \not\leq^* f_\alpha$. Then both families are fully unbounded.

(ii) We start with a model M of CH and iterate the dominating real poset along the well-founded poset $(\kappa \times \lambda, \preceq)$, where $(\alpha, \beta) \preceq (\gamma, \eta)$ iff $\alpha \leq \gamma$ and $\beta \leq \eta$. This is Hechler's model [H] for cofinally embedding $\kappa \times \lambda$ into $({}^\omega\omega, <^*)$. If $f_{(\alpha,\beta)}$ is the (α, β)th function added, then $(\alpha, \beta) \preceq (\gamma, \eta)$ and $(\alpha, \beta) \neq (\gamma, \eta)$ implies that $f_{(\alpha,\beta)} <^* f_{(\gamma,\eta)}$. Let $\mathscr{F} \subseteq {}^\omega\omega$ be of size δ. For each $f \in \mathscr{F}$ there is (α_f, β_f) s.t. $f \in M[G_{(\alpha_f,\beta_f)}]$. Fix α such that $\mathscr{F}^0 = \{f: \alpha_f = \alpha\}$ has size δ. Then there is a β above $\{\beta_f: f \in \mathscr{F}^0\}$. Therefore, $\mathscr{F}^0 \subseteq M[G_{(\alpha,\beta)}]$, which implies that $\mathscr{F}^0 <^* f_{(\alpha,\beta)}$. Therefore, $\mathscr{F}$ is not fully unbounded.

(iii) Start with a model of MA$+\mathfrak{c} = \kappa$ and add κ Cohen reals. It is straightforward to prove that $\mathfrak{b} = \aleph_1$, $\mathfrak{d} = \mathfrak{c} = \kappa$, and for any $\delta \leq \kappa$ uncountable the family consisting of the first δ Cohen reals is fully unbounded.

Theorem 3.3. *Let* $\mathscr{A} \subseteq [\omega_1]^\omega$ *be mad such that*

$$\forall x \in [\omega_1]^\omega \qquad |\mathscr{A} \restriction x| \geq \aleph_0 \rightarrow |\mathscr{A} \restriction x| = |\mathscr{A}| = \kappa.$$

Assume further that there exists an $\mathscr{F} \subseteq {}^\omega\omega$ *such that* $|\mathscr{F}| = \kappa$ *and* $\mathscr{F}$ *is fully unbounded; then* $\Psi(\mathscr{A})$ *is not countably metacompact.*

Proof. Fix $\mathscr{F}$ as in the hypothesis of the theorem and index $\mathscr{A}$ as $\{a_f: f \in \mathscr{F}\}$. For each $n, m \in \omega$, let $\mathscr{A}^n_m = \{a_f: f(n) = m\}$; then, for each n, $\mathscr{A} = \bigcup_{m<\omega} \mathscr{A}^n_m$ is a partition of $\mathscr{A}$. Assume $\Psi(\mathscr{A})$ is countably metacompact; then for each n there is $\langle U^n_m \rangle_{m<\omega} \subseteq [\omega_1]^{\omega_1}$ such that $\bigcap_{m<\omega} U^n_m = \varnothing$ and, $\forall m \forall k > m \forall a \in \mathscr{A}^n_k$, $a \subseteq^* U^n_m$.

For each n, choose $h(n)$ inductively such that $|\bigcap_{k\leq n} \omega_1 \backslash U^k_{h(k)}| = \aleph_1$. Clearly this can be done since $\forall n$ $\bigcup_{m<\omega} \omega_1 \backslash U^n_m = \omega_1$. Construct $x \in [\omega_1]^\omega$ such that $\forall k$ $x \subseteq^* \omega_1 \backslash U^k_{h(k)}$ and such that $|\mathscr{A} \restriction x| = |\mathscr{A}|$ as follows:

Let y_0 be a pseudo-intersection of $\{\omega_1 \backslash U^j_{h(j)}\}_{j<\omega}$ and pick $a_0 \in \mathscr{A}$, such that $|a_0 \cap y_0| = \aleph_0$. Having constructed $\{y_k: k < n\}$ and $\{a_k: k < n\}$, such that

(i) $y_k \subseteq U^k_{h(k)}$,
(ii) y_k is a pseudo-intersection of $\{\omega_1 \setminus U^j_{h(j)}\}_{j<\omega}$,
(iii) $|a_k \cap y_k| = \aleph_0$, and
(iv) $i \neq k$ implies $a_i \neq a_k$,

let y_n be a pseudo-intersection of $\{\omega_1 \setminus U^i_{h(i)}\}_{i<\omega}$ with $y_n \subseteq U^n_{h(n)}$ and such that every ordinal in y_n is above every ordinal in $\bigcup_{k<n} \alpha_k$. Pick $a_n \in \mathscr{A}$ such that $|a_n \cap y_n| = \aleph_0$. Let $x = \bigcup_{n<\omega} y_n$. Then $\forall k < \omega\ |a_k \cap x| = \aleph_0$. Therefore, $|\mathscr{A} \restriction x| \geq \aleph_0$; hence, by assumption $|\mathscr{A} \restriction x| = \kappa$.

Since $\mathscr{F}$ is fully unbounded, fix $f \in \mathscr{F}$ and $n \in \omega$ such that $|a_f \cap x| = \aleph_0$ and $h(n) < f(n)$. Therefore, $a_f \subseteq^* U^n_{h(n)}$, contradicting that $a_f \cap x$ is infinite and $x \cap U^n_{h(n)}$ is finite.

Corollary 3.4. *If $\mathfrak{c} = \aleph_2$ or if $\mathfrak{b}^+ = \mathfrak{c}$, then, for each $\mathscr{A} \subseteq [\omega_1]^\omega$ mad, $\Psi(\mathscr{A})$ is not countably metacompact.*

Proof. Let $\mathscr{A}$ be mad. Then, since $\mathfrak{b} \leq \mathfrak{a}$, either $\forall Y \in [\omega_1]^{\omega_1}\ |\mathscr{A} \restriction Y| = \mathfrak{c}$, or $\exists X \in [\omega_1]^{\omega_1}$ such that $\forall Y \in [X]^\omega\ |\mathscr{A} \restriction Y| = \mathfrak{b}$. In the first case Theorem 2.3 implies $\Psi(\mathscr{A})$ is not countably metacompact, while the second case follows from Theorem 3.3 and the existence of a fully unbounded family of size $\mathfrak{b}$.

Theorem 3.3 suggests the following question: Does the existence of a mad family of size κ imply the existence of a fully unbounded family of size κ? Since there are no fully unbounded families of regular size $> \mathfrak{d}$, the question is only interesting for mad families of singular cardinality or of size $\leq \mathfrak{d}$.

4. A CONSISTENT COUNTEREXAMPLE

We present in this section the construction of a regular, first countable, countably metacompact space X with a closed discrete subset which is not a G_δ. The space is constructed under the consistent assumption that $\mathfrak{b} = \omega_1$ and there exists a Q-set.

Definition 4.1. An uncountable subset of the reals is called a Q-set if every subset is a relative G_δ.

The following lemma was proved by Todorcevic (see [T, Lemma 2.5]). The set function H was used there to construct, among other things, a compact S-space from the assumption $\mathfrak{b} = \aleph_1$.

Lemma 4.2 (Todorcevic). *Assume $\mathfrak{b} = \omega_1$. Fix $Z \subseteq {}^\omega\omega$. There is a set function $H: Z \to [Z]^{\leq\omega}$ such that, for each $z \in Z$, $H(z)$ is either finite or a sequence converging to z with the property that, if Y and D are subsets of Z such that $Y \subseteq \overline{D}$ and Y is uncountable, then $\{y \in Y : H(y) \cap D$ is finite$\}$ is countable.*

Fix $Z \subseteq {}^\omega\omega$ a Q-set of size $\aleph_1$. We define a topology on $X = Z \times 2$ using H by letting $Z \times \{0\}$ be isolated in X, and a typical neighborhood of $(z, 1)$ looks like $\{(z, 1)\} \cup [H(z) \setminus F] \times \{0\}$ for some finite set F. Clearly X with this topology is first countable and regular. The fact that Z is a Q-set implies that the space is countably metacompact while the set function H assures that $Z \times \{1\}$ is not a G_δ in X.

Claim 4.3. *X is countably metacompact.*

Proof. Suppose $Z_0 \supseteq Z_1 \supseteq \cdots$ is a sequence of subsets of Z such that $\bigcap_{i<\omega} Z_i = \varnothing$. So $\{Z_i \times \{1\}: i < \omega\}$ is a typical decreasing sequence of closed sets in X. We use the fact that each Z_i is a G_δ in Z to construct the open fattenings U_i of $Z_i \times \{1\}$ with $\bigcap_{i<\omega} U_i = \varnothing$. For each $i < \omega$ there are Euclidean openings $V^i(n) \supseteq Z_i$ such that $\bigcap_{n<\omega} V^i(n) \cap Z = Z_i$. Without loss of generality we may assume that, for each n and each $i < j$, $V^j(n) \subseteq V^i(n)$. Then, for each $i < \omega$, $V^i(n) \times \{0\} \cup Z_i \times \{1\}$ is open in X.

Let $U_i = V^i(i) \times \{0\} \cup Z_i \times \{1\}$. It is straightforward to verify that $\bigcap_{i<\omega} U_i = \varnothing$.

Claim 4.4. *$Z \times \{1\}$ is not a G_δ in X.*

Proof. It clearly suffices to prove that if U is an open neighborhood of $Z \times \{1\}$ then $Z \times \{0\} \setminus U$ is countable.

Suppose $Y = \{z: (z, 0) \notin U\}$ is uncountable. Fix $D \subseteq Y$ countable dense in the Euclidean topology on Y. Then by Lemma 4.2 there is a $y \in Y$ such that $H(y) \cap D$ is infinite. Therefore $(y, 1) \in \overline{D \times \{0\}}$, which is a contradiction.

Unfortunately X is not normal. This follows from the next claim, that X is not countably paracompact, and from the fact that normal, countably metacompact spaces are countably paracompact.

Claim 4.5. *X is not countably paracompact.*

Proof. Let $\{X_n: n < \omega\}$ be a decreasing sequence of subsets of Z such that in the Euclidean topology, each X_n is $\aleph_1$-dense in Z. Let A be countable and dense in Z. Fix $n < \omega$ and $U_n \supseteq X_n \times \{1\}$ an open subset of X. By Lemma 4.2, $\{x \in X_n: H(x) \cap A \textit{ is finite}\}$ is countable. Therefore, $X'_n = \{x \in X_n: H(x) \cap A \textit{ is infinite}\}$ is $\aleph_1$-dense in Z. Letting $A' = A \times \{0\} \cap U_n$, Lemma 4.2 implies that $(X \times \{0\}) \cap \overline{U_n}$ is cocountable. Therefore, for any sequence of open sets $U_n \supseteq X_n \times \{1\}$, $\bigcap_{n<\omega} \overline{U_n} \neq \varnothing$.

In [FM] Fleissner and Miller prove the consistency of the existence of a Q-set concentrated on a countable set. As $\mathfrak{b} = \omega_1$ is equivalent to the existence of an uncountable set of reals concentrated on a countable set [R], $\mathfrak{b} = \omega_1$ and the existence of a Q-set are mutually consistent with ZFC.

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