

AN INFINITE LOOP SPACE MACHINE FOR THEORIES WITH NONCONTRACTIBLE MULTIPLICATION

IGOR KRIZ

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ABSTRACT. In this paper, we develop a new kind of infinite loop space machine starting with theories with possibly noncontractible multiplication. As an application, we give an example of an infinite loop space machine with ‘coherent’ homotopy inverses.

Schwänzl and Vogt [12] recently showed that if one wants to parametrize both the multiplication and the homotopy inverses in infinite loop spaces by the same ‘theory’ (for a definition, see §1.1), then the parametric spaces of the multiplication cannot be contractible. But the contractibility of these spaces is an intuitively motivated requirement, as it encodes the higher commutativities of the infinite loop space structure. Also, it is a property which made the passage from homotopy coherent to strict data possible in many cases (see, e.g., [11, 14]). One may therefore tend to conclude from [12] that homotopy inverses simply cannot be a part of the structure of an infinite loop space machine.

The purpose of this note, however, is to show that the last conclusion is, in effect, false. We construct an infinite loop space machine that accepts theories in which multiplication may be parametrized by a noncontractible space or there may even be no subspace specifically parametrizing multiplication (see Theorem 1.3). Instead, the theory must satisfy an alternate requirement (see (1.3.1)). The theories associated with E_∞ -operads satisfy both our requirement and the contractibility requirement [12, Example 2.4]. By a result of May and Thomason [8], in this case, our machine and the machines of Boardman and Vogt [1], May [4], and Segal [13] turn out equivalent spectra. Our machine, however, accepts theories of a more general nature, including theories which also encode homotopy inverses. By [12], of course, such theories cannot satisfy the contractibility requirement of [12, Definition 2.3].

As an application of our machine, we give a criterion for a self-functor on the category of based topological spaces to be weakly equivalent to Q ($QX = \Omega^\infty \Sigma^\infty X$; see Theorem 1.5).

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1. STATEMENT OF THE RESULTS

1.1. **Preliminaries.** Throughout this note, we shall work in the base category \mathcal{T} of nondegenerately based compactly generated weak Hausdorff spaces. One may also consider \mathcal{T} as a topological category where the object space is discrete and the homs are given the compact-open topology made compactly generated. We can then speak of *continuous* functors between \mathcal{T} and another topological category.

Equivalence means weak homotopy equivalence. We recall some basic facts about monads and theories, essentially following Boardman and Vogt [1].

The *theory of finite based sets* ϑ_S has object set \mathbf{N} consisting of the finite sets $\mathbf{n} = \{0, 1, \dots, n\}$ based at 0 ($n \geq 0$). Morphisms from \mathbf{m} to \mathbf{n} in ϑ_S are based maps from \mathbf{n} to \mathbf{m} (contravariantly). Note that a based space X specifies a functor $RX: \vartheta_S \rightarrow \mathcal{T}$ which is given on objects by $(RX)_n = X^n$ and on morphisms by the obvious maps (diagonals and inclusions of the base point). The category ϑ_S is isomorphic to the category Γ introduced by Segal [13]. The opposite category $(\vartheta_S)^{op}$ is denoted by \mathcal{F} [8].

A *based theory* is a topological category ϑ with object space \mathbf{N} and an inclusion $\vartheta_S \hookrightarrow \vartheta$ such that $\prod_{i=1}^n (\iota_i, -): \vartheta(\mathbf{m}, \mathbf{n}) \rightarrow \vartheta(\mathbf{m}, \mathbf{1})^n$ is the identity where $\iota_i \in \vartheta_S(\mathbf{n}, \mathbf{1})$ is the map in ϑ_S given by $\iota_i(1) = i$ and $\iota_i(0) = 0$.

A *ϑ -space* is a based space X together with a functor $A: \vartheta \rightarrow \mathcal{T}$ such that the composition $\vartheta_S \hookrightarrow \vartheta \xrightarrow{A} \mathcal{T}$ is equal to RX . Note that the category of ϑ_S -spaces is canonically isomorphic to \mathcal{T} ; in the sequel, these two categories will sometimes be identified.

Also recall that a *monad* in \mathcal{T} is a functor $D: \mathcal{T} \rightarrow \mathcal{T}$ together with natural transformations $\mu: DD \rightarrow D$ (composition) and $\eta: \text{Id} \rightarrow D$ (unit) such that the obvious associativity and unit diagrams commute (see, e.g., [4, Chapter 1]). A *D -space* is a based space together with an action map $\lambda: DX \rightarrow X$ such that the obvious associativity and left unit diagrams commute (see [4, Chapter 1]). A *D -functor*, on the other hand, is a functor $F: \mathcal{T} \rightarrow \text{Cat}$ where Cat is an arbitrary category together with a natural transformation $\rho: FD \rightarrow F$ (“right D -action”) such that, again, the obvious associativity and right unit diagrams commute.

Boardman and Vogt [1, Chapter 2] established passages both ways between monads theories. For a continuous monad D , we can talk about the *theory of D -spaces* ϑ_D where $\vartheta_D(\mathbf{m}, \mathbf{n}) = D(\mathbf{m})^n$. Thus, an element of $\vartheta_D(\mathbf{m}, \mathbf{n})$ is a mapping $\mathbf{n} \rightarrow D(\mathbf{m})$. This map and the monad composition μ give rise to a composition map

$$\vartheta_D(\mathbf{n}, \mathbf{p}) \times \vartheta_D(\mathbf{m}, \mathbf{n}) \rightarrow \vartheta_D(\mathbf{m}, \mathbf{p}).$$

Moreover, the identity $\text{Id} \in \vartheta_D(\mathbf{n}, \mathbf{n}) = D(\mathbf{n})^n$ has coordinates $\text{Id}_i = \eta_n(i)$ (recall that $\eta_X: X \rightarrow DX$ is the unit of the monad).

Conversely, to a theory ϑ , one can associate a monad D_ϑ . We first recall an auxiliary construction. For functors $X: \vartheta_S \rightarrow \mathcal{T}$ and $Y: \mathcal{F} \rightarrow \mathcal{T}$ we can define a ‘coend’

$$Y \times_{\vartheta_S} X = \coprod (Y_m \times X_m) / \sim,$$

where \sim is the equivalence relation generated by $(y\sigma, x) \sim (y, \sigma x)$

for $x \in X_m, y \in Y_n, \sigma \in \vartheta_S(\mathbf{m}, \mathbf{n})$. For a based theory ϑ and a ϑ_S -space X , put

$$(D_\vartheta X)_n = \vartheta(-, \mathbf{n}) \times_{\vartheta_S} X.$$

By Lemma 2.2, D_ϑ is a monad in ϑ_S -spaces (which is the same as a monad in \mathcal{F}).

Note that these definitions make sense more generally. For a general functor $D: \mathcal{F} \rightarrow \mathcal{F}$, not necessarily a monad, we define a functor $\vartheta_D: \mathcal{F} \rightarrow \mathcal{F}$ by

$$\vartheta_D(\mathbf{m}) = D(\mathbf{m}).$$

For a general functor $\vartheta: \mathcal{F} \rightarrow \mathcal{F}$ we define a functor $D_\vartheta: \mathcal{F} \rightarrow \mathcal{F}$ by

$$D_\vartheta X = \vartheta(-) \times_{\vartheta_S} X.$$

1.1.1. **Theorem** (Boardman-Vogt [1, Chapter 2]). *For a theory ϑ , $\vartheta_{D_\vartheta} \cong \vartheta$. Moreover, the categories of ϑ -spaces and D_ϑ -spaces are isomorphic via a canonical isomorphism which is the identity on objects.*

(The first sentence of this theorem is also true for an arbitrary functor $\vartheta: \mathcal{F} \rightarrow \mathcal{F}$).

Proof. The first statement follows from Lemma 2.1 below. For the second statement, note that $\vartheta(-, \mathbf{n}) \times_{\vartheta_S} -$ is left adjoint to the forgetful functor from ϑ -spaces to ϑ_S -spaces (it is the left Kan extension). Our statement therefore follows from the characterization of algebras over adjunction monads. \square

The relationship between D_{ϑ_D} and D for a functor D is more subtle. We can define a canonical map $\varepsilon: D_{\vartheta_D} \rightarrow D$ as follows: We have

$$D_{\vartheta_D}(X) = \coprod (D(\mathbf{n}) \times X^n) / \sim.$$

Define the map on each $D(\mathbf{n}) \times X^n$ as the adjoint

$$\varepsilon_n: D(\mathbf{n}) \times \mathcal{F}(\mathbf{n}, X) \rightarrow DX$$

of the obvious map $\nabla_n: \mathcal{F}(\mathbf{n}, X) \rightarrow \mathcal{F}(D\mathbf{n}, DX)$ (given just by applying the functor D).

But this brings up trouble: ε_n , and therefore ε , may not be continuous. For example, if $D: \mathcal{F} \rightarrow \mathcal{F}$ is defined by assigning to a topological space X the discrete set X , then $D_{\vartheta_D}(X) = X$ and

$$\varepsilon = \text{Id}: X \rightarrow \text{discrete } X.$$

The map ε_n is continuous if all the maps ∇_n are continuous which, in turn, happens if $D: \mathcal{F} \rightarrow \mathcal{F}$ is a continuous functor. (Recall that for a monad D , we had to assume that the maps ∇_n are continuous, anyway, to get a continuous composition on ϑ_D .) A functor D is called *finitary* if

(1.1.2) The map $\varepsilon: D_{\vartheta_D} \rightarrow D$ is an isomorphism of functors.

Since every functor of the form D_ϑ is obviously continuous, we see that every finitary functor is continuous.

1.1.3. **Example.** If \mathcal{C} is an operad, then the corresponding monad C is finitary.

We next state our results on connections between monads, theories, and infinite loop spaces. We recall the notion of an \mathcal{F} -space from May and Thomason [8]. First call a functor $X: \mathcal{F} \rightarrow \mathcal{T}$ *proper* if

(1.1.4) for any injection $\varphi \in \mathcal{F}(\mathbf{m}, \mathbf{n})$, $X(\varphi)$ is a Σ_φ -equivariant cofibration where Σ_φ is the group of permutations $\sigma: \mathbf{n} \rightarrow \mathbf{n}$ such that $\sigma(\text{Im } \varphi) = \text{Im } \varphi$.

An \mathcal{F} -space is a proper functor $X: \mathcal{F} \rightarrow \mathcal{T}$ such that

(1.1.5) X_0 is aspherical (i.e., equivalent to a point) and

(1.1.6) for $n > 1$, the map $X_n \rightarrow X_1^n$ with coordinates δ_i is an equivalence where $\delta_i: \mathbf{n} \rightarrow 1$ is the map in \mathcal{F} given by $\delta_i(j) = 1$ if $i = j$ and $\delta_i(j) = 0$ otherwise.

(Observe that δ_i may also be viewed as a map $\mathbf{1} \rightarrow \mathbf{n}$ in ϑ_S .) If we drop the properness requirement, such functors $X: \mathcal{F} \rightarrow \mathcal{T}$ are called *improper* \mathcal{F} -spaces. By [8, Appendix B], for every improper \mathcal{F} -space X there is a functorial \mathcal{F} -space X' together with a natural equivalence $X' \rightarrow X$.

1.2. **Theorem.** Let D be a monad in \mathcal{T} such that

(1.2.1) the canonical map $D(X \vee Y) \rightarrow DX \times DY$ is an equivalence for any pair of based spaces X, Y ,

(1.2.2) $D(*) \rightarrow *$ is an equivalence.

Then any D -space is canonically equivalent to an \mathcal{F} -space. (More precisely, there is a functorial \mathcal{F} -space \underline{X} and a natural equivalence $\underline{X}_1 \rightarrow X$.)

1.3. **Theorem.** Let ϑ be a theory such that

(1.3.1) $\prod_{i=1}^m (-, \delta_i): \vartheta(\mathbf{m}, \mathbf{1}) \rightarrow \vartheta(\mathbf{1}, \mathbf{1})^m$ is an equivalence for all $m \geq 0$.

Then any ϑ -space is canonically equivalent to an \mathcal{F} -space.

1.4. **Examples.** If \mathcal{D} is an E_∞ -operad, then the theory ϑ_D is an E_∞ -theory in the sense of Schwänzl and Vogt [12]. The theory ϑ_D also satisfies the hypothesis (1.3.1). In effect, the map of (1.3.1) is the canonical map $D(\mathbf{m}) \rightarrow D(\mathbf{1})^m$, which, if \mathcal{D} is an E_∞ -operad, is just a model for the canonical map

$$\coprod_{n_1 \cdots n_m \geq 0} B(\Sigma_{n_1} \times \cdots \times \Sigma_{n_m}) \rightarrow \left(\coprod_{n \geq 0} B(\Sigma_n) \right)^m,$$

which is easily seen to be an equivalence. (Here we use the convention $\Sigma_0 = \Sigma_1 = \{1\}$, which is, after all, consistent with the definition of the factorial.)

Thus, Boardman and Vogt [1] and Theorem 1.3 give two different ways of constructing a spectrum (or an infinite loop space) from the theory ϑ_D . Both ways are equivalent by the results of May and Thomason [8].

Consider now the monad Q ($QX = \Omega^\infty \Sigma^\infty X$). Q is not finitary, but it is known that the canonical map $D_{\vartheta_Q}(X) \rightarrow Q(X)$ is an equivalence, at least if the space X is sufficiently nice [13, 3]. The theory ϑ_Q satisfies (1.3.1), so we can reconstruct the infinite loop space structure from ϑ_Q -spaces by a conceptual procedure, without utilizing the “geometrical” structure of Q (Theorem 1.3). Moreover, ϑ_Q clearly contains homotopy inverses so that every ϑ_Q -space is

a grouplike \mathcal{F} -space, i.e., an infinite loop space. Note that, in accordance with the notation of Schwänzl and Vogt [12], there is also an augmentation $\vartheta_Q \rightarrow \mathcal{E}\mathcal{G}$ where $\mathcal{E}\mathcal{G}$ is the theory of abelian groups. Schwänzl and Vogt's theorem [12, Theorem 3.1] implies that the fibre $G^{-1}(\lambda_n)$, where λ_n is the n -fold multiplication, cannot be contractible. In effect, $G^{-1}(\lambda_n) \sim (Q_1)^n$ where Q_1 is the connected component of QS^0 consisting of maps of degree 1.

One may object that the theory ϑ_Q is not very useful, because it is too large. There is a nice geometrical model of ϑ_Q due to Caruso and Waner [2]. The functor \tilde{C}_n constructed in [2, pp. 150–151] has a natural structure of a monad. Moreover, one has stabilization maps $\tilde{C}_n \rightarrow \tilde{C}_{n+1}$ (a multiplication by I). Passage to a colimit (cofibration properties are good enough so we do not need to use a telescope) gives a finitary monad \tilde{C} together with an explicit equivalence $\tilde{C} \rightarrow Q$. The associated theory $\vartheta_{\tilde{C}}$ is the desired approximation of ϑ_Q . Theorem 1.2 suggests the question as to how general are the monads satisfying Condition 1.2.1. Even if we add the hypothesis that homotopy inverses exist, there are still examples very different from Q . We have, however, the following theorem. Call a functor $F: \mathcal{F} \rightarrow \mathcal{F}$ *proper* if $\vartheta_F: \mathcal{F} \rightarrow \mathcal{F}$ is proper.

1.5. Theorem. *A proper finitary functor $F: \mathcal{F} \rightarrow \mathcal{F}$ is weakly equivalent to Q if and only if it satisfies the following conditions:*

- (1.5.1) *The canonical map $F(X \vee Y) \rightarrow FX \times FY$ is an equivalence.*
- (1.5.2) *The ‘product’ $FX \times FX \leftarrow F(X \vee X) \xrightarrow{F\nabla} FX$ has a weak homotopy inverse map $FX \rightarrow FX$.*
- (1.5.3) *There is an element $\eta \in F(S^0)$ with the property that for each $k \geq 0$ there is an $n \geq 0$ such that if X is n -connected, then the map $H_*X \rightarrow H_*FX$ induced from the composition*

$$X = \mathcal{F}(S^0, X) \rightarrow \mathcal{F}(F(S^0), F(X)) \xrightarrow{\text{evaluation at } \eta} F(X)$$

is an isomorphism in degree $< n + k$.

2. PROOFS OF THE RESULTS

It will be clearer to assume, in this section, that the monads D from the statement of Theorem 1.2 turn cofibration maps into cofibration maps and that the functors $\mathcal{F} \rightarrow \mathcal{F}$ associated with the theories from the statement of Theorem 1.3 are proper. These hypotheses may be eliminated by the techniques of May [4], May and Thomason [8], and Segal [13].

We shall first need a few lemmas about functors $\vartheta: \mathcal{F} \rightarrow \mathcal{F}$ and the associated finitary functors D_ϑ .

2.1. Lemma. *For a functor $\vartheta: \mathcal{F} \rightarrow \mathcal{F}$, $D_\vartheta(\mathbf{m}) = \vartheta(\mathbf{m})$. \square*

2.2. Lemma. *Let, for functors $\vartheta', \vartheta'': \mathcal{F} \rightarrow \mathcal{F}$, $\vartheta = \vartheta' \times \vartheta''$. Then $D_\vartheta X = D_{\vartheta'} X \times D_{\vartheta''} X$.*

Proof. The map $D_\vartheta X \rightarrow D_{\vartheta'} X \times D_{\vartheta''} X$ is defined by

$$((t', t''), x) \mapsto (t', x) \times (t'', x)$$

where $t' \in \vartheta'(\mathbf{n})$, $t'' \in \vartheta''(\mathbf{n})$, and $x \in X^n$. The map in the opposite direction is defined by

$$(t', (x_1, \dots, x_n)) \times (t'', (y_1, \dots, y_m)) \mapsto ((t'\pi', t''\pi''), (x_1, \dots, x_n, y_1, \dots, y_m))$$

where $\pi' \in \vartheta_S(\mathbf{m} + \mathbf{n}, \mathbf{n})$ and $\pi'' \in \vartheta_S(\mathbf{m} + \mathbf{n}, \mathbf{m})$ are the projections to the first and last coordinates. The two maps are inverse to each other because of the identification by ϑ_S . \square

For $\vartheta \in \vartheta_S(\mathbf{n}, \mathbf{k})$, define $\vartheta \wr \mathbf{m} \in \vartheta_S(\mathbf{nm}, \mathbf{km})$ to operate on m -size blocks in the same way that φ does on points. More precisely, put

$$(\varphi \wr \mathbf{m})^{\text{Op}}((i - 1)m + j) = ((\varphi^{\text{Op}}(i) - 1)m + j) \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq m.$$

Let $\vartheta: \mathcal{F} \rightarrow \mathcal{S}$. Define $\underline{\vartheta}_m: \mathcal{F} \rightarrow \mathcal{S}$ by putting $\underline{\vartheta}_m(\mathbf{n}) = \vartheta(\mathbf{nm})$ and $\underline{\vartheta}_m(\varphi) = \vartheta(\varphi \wr \mathbf{m})$.

2.3. Lemma. $D_{\underline{\vartheta}_m}(X) \cong D_{\vartheta}(X \wedge \mathbf{m})$. (Note that $X \wedge \mathbf{m} = X \vee \dots \vee X$ (m times).)

Proof. A map $\sigma: D_{\vartheta}(X \wedge \mathbf{m}) \rightarrow D_{\underline{\vartheta}_m}(X)$ is defined as follows: Let

$$x = (x_1, \dots, x_n) \in (X \wedge \mathbf{m})^n, \quad x_i \neq *.$$

Define a map $\sigma_x: \mathbf{n} \rightarrow \mathbf{nm}$ by $\sigma_x(j) = (j - 1)m + i$ if $x_j \in i_i X$ where $i_i: X \rightarrow X \wedge \mathbf{m}$ is the i th injection. Let, further, $\nabla: X \wedge \mathbf{m} \rightarrow X$ be the codiagonal. For $d \in \vartheta_n$, let

$$\sigma: (d, x) \mapsto (d\sigma_x, \nabla^n x).$$

The inverse map $\tau: D_{\underline{\vartheta}_m}(X) \rightarrow D_{\vartheta}(X \wedge \mathbf{m})$ is defined as follows. Let $e \in \vartheta_{nm}$, $x = (x_1, \dots, x_n) \in X^n$. Put $\bar{x} = (i_1 x_1, \dots, i_m x_1, \dots, i_1 x_n, \dots, i_m x_n)$,

$$\tau: (e, x) \mapsto (e, \bar{x}).$$

Define a map $\lambda \in \vartheta_S(\mathbf{n}, \mathbf{nm})$ by $\lambda((i - 1)m + j) = i$, $1 \leq i \leq m$, $1 \leq j \leq n$. Compute

$$\begin{aligned} \tau\sigma(d, x) &= (d\sigma_x, i_1 \nabla x_1, \dots, i_m \nabla x_1, \dots, i_1 \nabla x_n, \dots, i_m \nabla x_n) \sim (d, x), \\ \sigma\tau(e, x) &= (e\sigma_{\bar{x}}, x_1, \dots, x_1, \dots, x_n, \dots, x_n) \\ &= (e\sigma_{\bar{x}}, \lambda(x_1, \dots, x_n)) \sim (e\sigma_{\bar{x}}(\lambda \wr \mathbf{m}), x) = (e, x) \end{aligned}$$

(as $\sigma_{\bar{x}}(\lambda \wr \mathbf{m}) = 1_n$ in ϑ_S). \square

2.4. Proof of Theorem 1.2. Put, for $X \in \text{Obj } \mathcal{S}$, $\underline{D}_n X = D(X \wedge \mathbf{n})$. Then $\underline{D}X$ is an \mathcal{F} -space via the action of \mathcal{F} on the \mathbf{n} -coordinate. To see that $D(X \wedge \mathbf{n}) \rightarrow (DX)^n$ is an equivalence, simply note that $X \wedge \mathbf{n} = X \vee \dots \vee X$ (n times) and use condition (1.2.1). Now note that $\underline{D}_m: \mathcal{S} \rightarrow \mathcal{S}$ is a D -functor. To define the action $\underline{D}_m DX \rightarrow \underline{D}_m X$, first consider an injection $i_i: X \rightarrow X \wedge \mathbf{m}$. This injection induces a map $D(i_i): D(X) \rightarrow D(X \wedge \mathbf{m})$. Summing these maps gives a map

$$D(i_1) \vee \dots \vee D(i_m): D(X) \wedge \mathbf{m} \rightarrow D(X \wedge \mathbf{m}).$$

We now define the action $\underline{D}_m DX \rightarrow \underline{D}_m X$ to be the composition

$$\underline{D}_m DX = D(DX \wedge \mathbf{m}) \xrightarrow{D(Di_1 \vee \dots \vee Di_m)} DD(X \wedge \mathbf{m}) \rightarrow D(X \wedge \mathbf{m}) = \underline{D}_m X.$$

The proof of the associativity of the action is easily reduced to the commutativity of the diagram:

$$\begin{array}{ccc}
 DD\mathbf{X} \wedge \mathbf{m} & \xrightarrow{D_{i_1} D \vee \dots \vee D_{i_m} D} & D(D\mathbf{X} \wedge \mathbf{m}) \\
 \searrow^{DD_{i_1} \vee \dots \vee DD_{i_m}} & & \downarrow^{D(D_{i_1} \vee \dots \vee D_{i_m})} \\
 & & DD(\mathbf{X} \wedge \mathbf{m})
 \end{array}$$

This diagram commutes when precomposed with either of the injections $i_i DD$ be the functoriality of D and therefore commutes.

Now, since each $\underline{D}_m: \mathcal{F} \rightarrow \mathcal{F}$ is a D -functor, $(\underline{D}: \mathcal{F} \rightarrow \mathcal{F})$ -spaces is a D -functor. Let, for a D -space X ,

$$\underline{X} = B(\underline{D}, D, X).$$

Then \underline{X} is an \mathcal{F} -space, and the collapse map

$$\underline{X}_1 = B(D, D, X) \rightarrow X$$

is an equivalence. Moreover, $B(\underline{D}, D, X)$ is an \mathcal{F} -space since it is the geometric realization of a simplicial \mathcal{F} -space. \square

We now turn to the proof of Theorem 1.3. Note that, in view of the proof of Theorem 1.2, it would be sufficient to show that (1.3.1) implies that $D_{\vartheta}(X \wedge \mathbf{m}) \rightarrow (D_{\vartheta}X)^m$ is an equivalence. However, we do not know how to prove this. By Lemmas 2.2 and 2.3, it would be sufficient to know that if $\vartheta \rightarrow \vartheta'$ is an equivalence of functors $\mathcal{F} \rightarrow \mathcal{F}$ then the induced map $D_{\vartheta}X \rightarrow D_{\vartheta'}X$ is an equivalence. This, however, is false; the issue is the lack of a freeness condition on the actions of $\Sigma_n \subseteq \mathcal{F}$.

To prove Theorem 1.3, we use simplicial sets. Recall (from May [7]) that the singular set functor S from spaces to simplicial sets is right adjoint to the realization functor T . Analogously, S and T are functors between based spaces and simplicial based sets and are still adjoint (because the unit and counit are based maps). On occasion, we shall also use T to denote the realization of simplicial spaces (in the cases when the symbol $||$ is too awkward).

2.5. Lemma. *Let $\vartheta_1 \rightarrow \vartheta_2$ be an equivalence of functors $\mathcal{F} \rightarrow \mathcal{F}$. Then, for any based set S , the induced map*

$$D_{\vartheta_1}S \rightarrow D_{\vartheta_2}S$$

is an equivalence.

Proof. For a compact space K , a map $\varphi: K \rightarrow D_{\vartheta_1}S$ factors through $D_{\vartheta_1}L$ for some finite subset $L \subseteq S$. The map

$$D_{\vartheta_1}L \rightarrow D_{\vartheta_2}L$$

is an equivalence by Lemma 2.1 and the hypothesis. \square

2.6. Lemma. *For a simplicial space A , $D_{\vartheta}TA = T(D_{\vartheta})_*(A)$.*

Proof. Compute

$$D_{\vartheta}TA = \theta \times_{\theta_S} RTA \cong \theta \times_{\theta_S} TR_*A \cong T(\theta \times_{\theta_S} R_*A) = T(D_{\vartheta})_*(A). \quad \square$$

2.7. **Lemma.** *Let $\vartheta: \mathcal{F} \rightarrow \mathcal{G}$ be a proper functor. Then, for any simplicial set A , $\underline{D}_\vartheta TA$ is an \mathcal{F} -space.*

Proof. Lemma 2.3 asserts that $\underline{D}_{\vartheta_n} = D_{\vartheta_n}$, and Lemma 2.2 asserts that $D_{(\vartheta^n)} = (D_\vartheta)^n$. We need to show that the map

$$D_{\vartheta_n} TA \rightarrow (D_\vartheta)^n TA$$

(as in the first paragraph of 2.4) is an equivalence. Using Lemma 2.6, all we need to show is that

$$TD_{\vartheta_n} A \xrightarrow{\sim} T((D_\vartheta)^n A).$$

But $D_{\vartheta_n} A \rightarrow (D_\vartheta)^n A$ is a simplicial equivalence by Lemma 2.5. \square

2.8. *Proof of Theorem 1.3.* By Lemma 2.7, for a ϑ -space X we may put

$$X = B(D_\vartheta T, SD_\vartheta T, SX). \quad \square$$

We now turn to the proof of Theorem 1.5. There is a map

$$\lambda: \Sigma^n FX \rightarrow F\Sigma^n X$$

given as follows: For $s \in S^n$, $a \in (\theta_F)_n$, and $x = (x_1, \dots, x_n) \in X^n$, put

$$\lambda(s \wedge (a, x)) = (a, (s \wedge x_1, \dots, s \wedge x_n))$$

(observe that the base points are OK, since choosing $s = 0$ or $x = *$ gives $*$ on both sides).

2.9. **Lemma.** *Under the assumptions (1.5.1) and (1.5.2) of Theorem 1.5, the composition*

$$FX \xrightarrow{\eta^F} \Omega^n \Sigma^n FX \xrightarrow{\Omega^n \lambda} \Omega^n F\Sigma^n X$$

is an equivalence.

Proof. Put $\vartheta = \vartheta_F$. By Lemma 2.3,

$$D_{\vartheta_m} X \cong F(X \wedge \mathbf{m}),$$

and therefore, by condition (1.5.1), $D_{\vartheta_{(-)}} X$ is an \mathcal{F} -space. Segal gives a canonical map

$$\Sigma^n FX \rightarrow B_n FX = D_{\vartheta_{(-)}} X \times_{\theta_S} R(S^n)$$

(see [13, Proposition 3.2]) whose adjoint

$$FX \rightarrow \Omega^n \Sigma^n FX \rightarrow \Omega^n (D_{\vartheta_{(-)}} X \times_{\theta_S} R(S^n))$$

is an equivalence (by condition (1.5.2) and [13, Proposition 1.5]). Now, by [13, Lemma 3.7], $D_{\vartheta_{(-)}} X \times_{\theta_S} R(S^n)$ is canonically identified with $F\Sigma^n X$, with a commutative diagram of canonical maps:

$$\begin{array}{ccc} \Sigma^n FX & & \\ \downarrow & \searrow \lambda & \\ D_{\vartheta_{(-)}} X \times_{\theta_S} R(S^n) & = & F\Sigma^n X \end{array}$$

Our statement follows. \square

2.10. *Proof of Theorem 1.5.* Necessity is obvious, as the conditions (1.5.1)–(1.5.3) are satisfied by Q (even though Q is not a finitary functor). Conversely, let (1.5.1)–(1.5.3) hold. Denote the natural transformation $(\eta, -): I_d \rightarrow F$ induced by the element $\eta \in (\theta_F)_1$ by u . Note that the following diagrams commute:

$$(2.10.1) \quad \begin{array}{ccc} \Sigma^i X & \xrightarrow{\Sigma^i u} & \Sigma^i F X & & (s \wedge x) \mapsto (s \wedge (\eta, x)) \\ & \searrow u \Sigma^i & \downarrow \lambda & & \downarrow \\ & & F \Sigma^i X & & (\eta, s \wedge x) \end{array}$$

$$(2.10.2) \quad \begin{array}{ccc} \Sigma^{i+j} F X & \xrightarrow{\Sigma^i \lambda} & \Sigma^i F \Sigma^j X \\ & \searrow \lambda & \downarrow \lambda \Sigma^j \\ & & F \Sigma^{i+j} X \end{array}$$

$$(s^i \wedge s^j \wedge (a_k, (x_1, \dots, x_k))) \mapsto (s^i \wedge (a_k, s^j \wedge x_1, \dots, s^j \wedge x_k))$$

$$\searrow \quad \downarrow$$

$$(a_k, (s^i \wedge s^j \wedge x_1), \dots, (s^i \wedge s^j \wedge x_k))$$

On the element level, our notation is $s^i \in S^i$, $s^j \in S^j$, $a \in (\theta_F)_k$, and $x = (x_1, \dots, x_k) \in X^k$. Now by Lemma 2.9, we have a map

$$\Omega^n \Sigma^n X \xrightarrow{\Omega^n u \Sigma^n} \Omega^n F \Sigma^n X \xleftarrow{\sim} F X.$$

Moreover, these maps stabilize via a diagram

$$\begin{array}{ccccc} F X & \xrightarrow{\sim} & \Omega^n F \Sigma^n X & \xleftarrow{\Omega^n u \Sigma^n} & \Omega^n \Sigma^n X \\ & & \Omega^n \eta F \Sigma^n \downarrow & & \downarrow \Omega^n \eta \Sigma^n \\ & & \Omega^{n+1} \Sigma F \Sigma^n X & \xleftarrow{\Omega^{n+1} \Sigma u \Sigma^n} & \Omega^{n+1} \Sigma^{n+1} X \\ & & \downarrow \Omega^{n+1} \lambda \Sigma^n X & & \swarrow \Omega^{n+1} u \Sigma^{n+1} \\ & & \Omega^{n+1} F \Sigma \Sigma^n X & & \end{array}$$

The top rectangle commutes by the naturality of the transformation $\Omega^n \eta: \Omega^n \rightarrow \Omega^{n+1} \Sigma$. The lower triangle commutes by the commutativity of (2.10.1) for $i = 1$. The left-hand triangle is adjoint to (2.10.2) for $i = 1, j = n$.

We now conclude our proof by showing that $\Omega^n u \Sigma^n$ is a k -equivalence for sufficiently large n . Diagram (2.10.1) ($i = n$) by adjunction gives rise to a

commutative diagram:

$$\begin{array}{ccc}
 \Sigma^m X & \xrightarrow{u\Sigma^m} & F\Sigma^m X \\
 \eta\Sigma^m \downarrow & & \downarrow \\
 \Omega^n \Sigma^n \Sigma^m X & \xrightarrow{\Omega_n u\Sigma_{n+m}} & \Omega^n F\Sigma^n \Sigma^m X
 \end{array}$$

By assumption 1.5.3, if m is sufficiently large, the top row induces an isomorphism in homology in degrees $\leq m+k$. The right-hand column is an equivalence by Lemma 2.9. The left-hand column is an isomorphism in degrees $\leq m+k$ if n is sufficiently large. We conclude that if m, n are sufficiently large, the bottom row is an $(m+k)$ -equivalence, and, therefore, $\Omega^{m+n} u\Sigma^{m+n}: \Omega^{m+n} \Sigma^{m+n} X \rightarrow \Omega^{m+n} F\Sigma^{m+n} X$ is a k -equivalence. \square

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