ON THE METRICAL THEORY OF CONTINUED FRACTIONS

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Abstract. Suppose $b_k$ denotes either $\phi(k)$ or $\phi(p_k)$ ($k = 1, 2, \ldots$) where the polynomial $\phi$ maps $\mathbb{N}$ to $\mathbb{N}$ and $p_k$ denotes the $k$th rational prime. Suppose $(c_k(x))_{k=1}^{\infty}$ denotes the sequences of partial quotients of the continued function expansion of the real number $x$. Then for certain functions $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ we show that

$$\lim_{N \to \infty} \frac{F(c_{b_1}(x)) + \cdots + F(c_{b_k}(x))}{N} = \frac{1}{(\log 2)} \int_0^1 \frac{F(c_1(x))}{1 + x} \, dx$$

almost everywhere with respect to Lebesgue measure. This result with $b_k = k$ is classical and due to Ryll-Nardzewski.

In this note we consider the continued fraction expansion of a real number $x$:

$$x = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \cdots}}.$$

Suppose $(X, \beta, \mu, T)$ is a measure-preserving dynamical system. Following [1] we say a strictly increasing sequence of nonnegative integers $(b_n)_{n=1}^{\infty}$ is $L^p$-good universal if for each function $f$ in $L^p(X, \beta, \mu)$

$$\lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{b_n} x) = l(x),$$

exists $\mu$ almost everywhere.

We prove

Theorem 1. Suppose the function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous, increasing, and that for some $p \geq 1$ we have

$$\int_0^1 \frac{|F(c_1(x))|^p}{x + 1} \, dx < \infty.$$

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For each \( n \) in \( \mathbb{N} \) and arbitrary nonnegative real numbers \( a_1, \ldots, a_n \) we define
\[
M_{F,n}(a_1, \ldots, a_n) = F^{-1}\left[ \frac{F(a_1) + \cdots + F(a_n)}{n} \right].
\]
Assume \( (b_n)_{n=1}^{\infty} \) is any strictly increasing sequence of nonnegative integers satisfying the following two conditions:

(i) For each irrational number \( \alpha \), if \( (x) \) denotes the fractional part of \( x \), then \( ((b_n \alpha))_{n=1}^{\infty} \) is uniformly distributed modulo one.

(ii) The sequence \( (b_n)_{n=1}^{\infty} \) is \( L^p \)-good universal.

Then
\[
\lim_{n \to \infty} M_{F,n}(c_{b_1}(x), \ldots, c_{b_n}(x)) = F^{-1}\left[ \frac{1}{\log 2} \int_0^1 \frac{F(c_1(x))}{1+x} \, dx \right]
\]
almost everywhere with respect to Lebesgue measure.

The special case \( b_n = n \) of this result is due to Ryll-Nardzewski [9] who unified earlier work of Khinchine [4]. More general examples of sequences \( (b_n)_{n=1}^{\infty} \) satisfying (i) and (ii) include, if \( \phi \) denotes an arbitrary nonconstant polynomial mapping \( \mathbb{N} \) to itself, \( b_n = \phi(n) \) and \( b_n = \phi(p_n) \). Here \( p_n \) denotes the \( n \)th rational prime. That both these examples satisfy (i) is well known [11, 8]. That they satisfy (ii) for any \( p > 1 \) is shown in [2, 6].

We mention some applications of Theorem 1 which follow for appropriate choice of \( F \).

**Corollary 2.** If the sequence \( (b_n)_{n=1}^{\infty} \) satisfies (i) and (ii) then
\[
\lim_{n \to \infty} \frac{1}{N} \sum_{k=1}^{N} c_{b_k}(x) = \infty
\]
almost everywhere with respect to Lebesgue measure.

**Corollary 3.** If the sequence \( (b_n)_{n=1}^{\infty} \) satisfies (i) and (ii) then
\[
\lim_{N \to \infty} (c_{b_1}(x) \cdots c_{b_N}(x))^{1/N} = \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k^2 + 2k} \right)^{(\log k)/\log 2}
\]
almost everywhere with respect to Lebesgue measure.

**Corollary 4.** Suppose the sequence \( (b_n)_{n=1}^{\infty} \) satisfies (i) and (ii). Let \( P_N(x, q) \) \((N = 1, 2, \ldots)\) denote the number of \( c_{b_1}(x), \ldots, c_{b_N}(x) \) such that if \( c_{b_i}(x) = q \) then
\[
\lim_{N \to \infty} \frac{P_N(x, q)}{N} = \frac{1}{(\log 2)} \log \frac{(q + 1)^2}{q(q + 2)}
\]
almost everywhere with respect to Lebesgue measure.

We remark that Corollary 2 follows by applying Theorem 1 to \( F(x) = x \). Strictly speaking this function does not satisfy the hypothesis of Theorem 1, however, setting \( F(x) = \min(f(x), M) \) and letting \( M \to \infty \) overcomes this difficulty. See the argument leading to (7) for more details. Corollaries 3 and 4 are straightforward.

All these corollaries are known in the case \( b_n = n \). In this special case
Corollaries 2 and 3 are due to Khinchine [4] and Corollary 4 is due to Lévi [5].
We say a transformation \( T \) of a probability space \((X, \beta, \mu)\) is weakly mixing if for each pair of sets \( A \) and \( B \) in the sigma-algebra \( \beta \) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} |\mu(A \cap T^{-k}B) - \mu(A)\mu(B)| = 0.
\]

Let \( U \) be the Koopman unitary operator on \( L^2(X, \beta, \mu) \) defined pointwise by \( f(x) \to f(Tx) \). We remind the reader that by approximating arbitrary \( L^2 \) functions \( f \) and \( g \) by linear combinations of step functions we see that (1) is equivalent to the formally stronger statement

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \langle (U^k f, g) - \langle f, 1 \rangle \langle g, 1 \rangle \rangle = 0.
\]

Here \( \langle , \rangle \) denotes the standard inner product on \( L^2(X, \beta, \mu) \).

To prove Theorem 1 we need some lemmas.

**Lemma 5.** Suppose \( (f_n)_{n=1}^{\infty} \) is a bounded nonnegative sequence of real numbers. Then the following two statements are equivalent:

(i) \( \sum_{n=1}^{N} f_n = o(N) \) and

(ii) \( \sum_{n=1}^{N} f_n^2 = o(N) \).

**Proof.** In [7] following Koopman and von Neumann it is shown that (i) is equivalent to the statement that there exists \( M \subseteq \mathbb{N} \) such that

\[
\lim_{N \to \infty} \frac{\#(M \cap [1, N])}{N} = 1 \quad \text{and} \quad \lim_{n \to \infty} f_n = 0.
\]

The equivalence of (i) and (ii) follows immediately from this. □

We say a sequence \( (a_n)_{n=-\infty}^{\infty} \) is positive definite if for any sequence \( \{z_n\}_{n=-\infty}^{\infty} \) having only finitely many nonzero terms we have

\[
\sum_{n, m} a_{n-m} z_n \overline{z_m} \geq 0.
\]

We have the Herglotz theorem [3, p. 38].

**Lemma 6.** A sequence \( (a_n)_{n=-\infty}^{\infty} \) is positive definite if and only if there exists a positive measure \( \mu \) on \( \mathbb{T} \) such that

\[
a_n = \int_{\mathbb{T}} e^{-int} \, d\mu \quad (n \in \mathbb{Z})
\]

(i.e., the \( n \)th Fourier coefficient of \( \mu \) denoted by \( \hat{\mu}(n) \)).

We now have Wiener's criteria [3, p. 42] for the continuity of a measure.

**Lemma 7.** A measure \( \mu \) on \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \) has no atoms if and only if

\[
\lim_{n \to \infty} \frac{1}{2n + 1} \sum_{-N}^{N} |\hat{\mu}(n)|^2 = 0.
\]

**Lemma 8.** Suppose that the measure-preserving dynamical system \((X, \beta, \mu, T)\) is weakly mixing and that the strictly increasing sequence of nonnegative integers
finite. Then if \( f \in L^p(X, \beta, \mu) \) with \( p > 1 \) we have \( I(x) = \int_X f \, d\mu \) \( \mu \) almost everywhere.

**Proof.** We begin by proving the statement in norm, that is,

\[
\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{k=1}^{N} f(T^{b_k} x) - \int_X f \, d\mu \right\| = 0,
\]

where \( \|f\| \) denotes the \( L^2(X, \beta, \mu) \) norm of the function \( f \). By arguing with \( f(x) - \int_X f \, d\mu \) instead of \( f(x) \) if necessary, we can assume without loss of generality that \( \int_X f \, d\mu = 0 \). Now let \( U \) denote the Koopman linear isometry on \( L^2(X, \beta, \mu) \). As is immediately verified, letting \( U^n \) denote \( (U^*)^n \) if \( n < 0 \), the sequence \( (U^n f, f) \) is positive definite; hence, there is a measure \( \omega_f \) on \( T \), dependent on \( f \) such that

\[
(U^n f, f) = \int_T e^{-int} d\omega_f(t) \quad (n \in \mathbb{Z}).
\]

Some basic properties of \( U \) can be found in [10, p. 25]. This means that

\[
\left\| \frac{1}{N+1} \sum_{k=0}^{N} f(T^{b_k} x) \right\|^2 = \frac{1}{(N+1)^2} \sum_{0 \leq k_1, k_2 \leq N} (U^{b_{k_2}-b_{k_1}} f, f)
\]

\[
= \frac{1}{(N+1)^2} \sum_{0 \leq k_1, k_2 \leq N} \int_0^1 e^{2\pi i (b_{k_2}-b_{k_1}) \theta} d\omega_f(\theta)
\]

\[
= \int_0^1 \left\| \frac{1}{N+1} \sum_{k=0}^{N} e^{2\pi i b_{k} \theta} \right\|^2 d\omega_f(\theta).
\]

By Lemma 7 the measure \( \omega_f \) is nonatomic if we can show that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} |\omega_f(n)|^2 = 0.
\]

Because \( |\omega_f(n)| \leq \|\omega_f\| \) for each \( n \in \mathbb{Z} \), by Lemma 5, it is sufficient to show

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} |\omega_f(n)| = 0.
\]

Now note that the weak-mixing of \( T \) in the form (2) and the fact that \( \int_X f \, d\mu = 0 \) give

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} |\omega_f(n)| = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} |(U^k f, f)| = 0.
\]

This means \( \omega_f \) is nonatomic. It now follows that (3) holds using (4) and (i), that is, the fact that \( \{(b_k \theta)\}_{k=1}^{\infty} \) is uniformly distributed modulo one for each irrational \( \theta \).

To prove the almost everywhere convergence we argue as follows. Let \( (N_t)_{t=1}^{\infty} \) denote a strictly increasing sequence of integers such that

\[
\left\| \frac{1}{N_t} \sum_{k=1}^{N_t} f(T^{b_k} x) - \int_X f \, d\mu \right\| < \frac{1}{t}.
\]
Thus
\[ \sum_{t=1}^{\infty} \left\| \frac{1}{N_t} \sum_{k=1}^{N_t} f(T^{b_k}x) - \int_x f \, d\mu \right\|^2 < \infty. \]

Hence rearranging using Fatou's lemma we have
\[ \int_x \left( \sum_{t=1}^{\infty} \left\| \frac{1}{N_t} \sum_{k=1}^{N_t} f(T^{b_k}x) - \int_x f \, d\mu \right\|^2 \right) \, d\mu < \infty. \]

Thus using the fact that if \( \sum a_n < \infty \) with \( a_n \geq 0 \) then \( a_n = o(1) \), we have
\[ \lim_{t \to \infty} \frac{1}{N_t} \sum_{k=1}^{N_t} f(T^{b_k}x) = \int_x f \, d\mu \quad \text{a.e.} \tag{5} \]

By (ii), that is, the \( L^p \)-good universality property, the limit exists and so it must be \( \int_x f \, d\mu \) as required by (5). \( \Box \)

We now complete the proof of Theorem 1. Let \( T \) denote the map of \( (0, 1) \) defined by \( Tx = (1/x) \). We note that the partial quotients \( (c_k(x))_{k=1}^{\infty} \) of \( x \) satisfy \( c_k(x) = c_1(T^{k-1}x) \). Gauss observed that the map \( T \) preserves the measure
\[ \mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}. \tag{6} \]

This measure is equivalent to Lebesgue measure on \([0, 1]\) (in the sense that it has the same null sets). In addition the transformation \( T \) is weak mixing on \([0, 1]\) with respect to the measure \( \mu \) in (6). If \( \int_0^1 F(c_1(x)) \, dx/(1+x) \) is finite then Theorem 1 follows immediately from Lemma 8. If \( \int_0^1 F(c_1(x)) \, dx/(1+x) \) is not finite, we argue as follows. Let
\[ f_M(x) = \begin{cases} F(c_1(x)) & \text{if } |F(c_1(x))| \leq M, \\ M & \text{if } |F(c_1(x))| > M. \end{cases} \]

This means that for each \( M \geq |F(1)| \) and for almost all \( x \)
\[ \lim_{N \to \infty} \inf \frac{1}{N} \sum_{k=1}^{N} F(c_k(x)) \geq \lim_{N \to \infty} \inf \frac{1}{N} \sum_{k=1}^{N} f_M(T^{b_k-1}x) \]
\[ = \frac{1}{(\log 2)} \int_0^1 \frac{f_M(x)}{1+x} \, dx. \tag{7} \]

Now letting \( M \) tend to infinity completes the proof.

**References**


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