THE ASSOCIATED GRADED RING
AND THE INDEX OF A GORENSTEIN LOCAL RING

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(Communicated by Maurice Auslander)

Abstract. Let \((R, \mathfrak{m}, k)\) be a Gorenstein local ring. It is shown that if the associated graded ring \(G(R)\) of \(R\) is Cohen-Macaulay, then the index of \(R\) is equal to the generalized Loewy length of \(R\).

Introduction

Let \((R, \mathfrak{m}, k)\) be a commutative noetherian Gorenstein local ring. Associated with a finitely generated \(R\)-module \(M\), we have Auslander's \(\delta\)-invariant \(\delta_R(M)\) of \(M\). It is defined to be the smallest integer \(n\) such that there is an epimorphism \(X \oplus R^n \to M\) with \(X\) a maximal Cohen-Macaulay module with no free summands. Of particular interest is the \(\delta\)-invariant of \(R/\mathfrak{m}^n\). We know that \(\delta_R(R/\mathfrak{m}^n) \leq 1\) and eventually equals to 1 [3]. The smallest \(n\) such that \(\delta_R(R/\mathfrak{m}^n) = 1\) is called the index of \(R\). One of the main questions is how the index is related to other invariants of \(R\).

Let \(N\) be a module of finite length. The Loewy length of \(N\), denoted by \(ll(N)\), is the smallest integer \(n\) such that \(\mathfrak{m}^nN = 0\). When \(R\) is 0-dimensional, the index of \(R\) is the same as the Loewy length of \(R\). For \(R\) of positive dimension, the generalized Loewy length of \(R\) is the minimum of all integers \(ll(R/(x))\), where \(x\) is a system of parameters (sop) of \(R\). In [4] we put forth the following

Conjecture. Let \(R\) be a Gorenstein local ring. Then the index of \(R\) is equal to the generalized Loewy length of \(R\).

The conjecture was shown in the affirmative for hypersurface rings [4] and for homogeneous Gorenstein \(k\)-algebras [5]. (Here one has to extend the above concepts in an obvious way to homogeneous \(k\)-algebras.) The main result of this paper is to show that the conjecture holds for \(R\) if the associated graded ring \(G(R)\) of \(R\) is Cohen-Macaulay. It is known that the associated graded rings of hypersurface rings and homogeneous Gorenstein \(k\)-algebras are Gorenstein. As a consequence, we obtain the earlier results. Moreover, much work has been done on the Cohen-Macaulayness of \(G(R)\) of a Gorenstein local ring \(R\); we thus obtain a larger class of Gorenstein rings for which the conjecture hold.

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We recall some basic facts on \( \delta \)-invariant of a module in \( \S 1 \). Section 2 is devoted to proving the main result.

1. Some preliminaries

In this paper \((R, m, k)\) will always be a Gorenstein local ring, and all \( R \)-modules are assumed to be finitely generated. Since both the \( \delta \)-invariant and the generalized Loewy length are invariants under faithfully flat local ring extensions, we may assume that \( k \) is an infinite field and \( R \) is complete, whenever necessary. Let \( G(R) = R/m \oplus m/m^2 \oplus m^2/m^3 \oplus \cdots \) be the associated graded ring. We denote the map \( R \to G(R) \) which takes each element \( x \) of \( R \) to its initial form in \( G(R) \) by "\( - \)". The following are some basic facts about the \( \delta \)-invariant of a module over \( R \).

**Lemma 1.1** [1, 3]. Let \( M \) and \( N \) be \( R \)-modules. Then:

(a) If \( N \) is an epimorphic image of \( M \), then \( \delta_R(M) \geq \delta_R(N) \).

(b) \( \delta_R(M \oplus N) = \delta_R(M) + \delta_R(N) \).

**Lemma 1.2** [2]. Let \( M \) be an \( R \)-module and \( x \in m \) be regular on both \( R \) and \( M \). Set \( \overline{R} = R/xR \). Then \( \delta_R(M) = \delta_{\overline{R}}(M/xM) \).

**Lemma 1.3** [1]. For any integer \( s \geq 1 \), we have \( \delta_R(m^s) = 0 \).

**Lemma 1.4** [4]. Let \( R \) be a 0-dimensional Gorenstein local ring and \( I \) an ideal of \( R \). Then \( \delta_R(R/I) \neq 0 \) if and only if \( I = (0) \). In particular, \( \text{index}(R) = ll(R) \).

Let \( x \in m \) be \( R \)-regular. Set \( \overline{R} = R/xR \). An \( \overline{R} \)-module \( N \) is said to be weakly liftable to \( R \) if there exists an \( R \)-module \( L \) on which \( x \) is regular such that \( N \) is isomorphic to a direct summand of the module \( L/xL \) [2]. The following result is useful.

**Lemma 1.5** [2]. For an \( \overline{R} \)-module \( N \) the following are equivalent.

(a) \( N \) is weakly liftable to \( R \).

(b) \( \Omega_R(N)/x\Omega_R(N) \cong N \oplus \Omega_{\overline{R}}(N) \), where \( \Omega_R(N) \) (resp. \( \Omega_{\overline{R}}(N) \)) is the first syzygy of \( N \) over \( R \) (resp. \( \overline{R} \)).

The weak liftability and the \( \delta \)-invariant of a module are closely related. For further details see [2].

2. The main result

Let \((R, m, k)\) be a Gorenstein local ring. The index of \( R \) is defined to be the smallest integer \( n > 0 \) such that \( \delta(R/m^n) \neq 0 \). The generalized Loewy length of \( R \) is the minimum of all integers \( ll(R/(x)) \), where \( x \) is a sop of \( R \). Our main result in this paper is the following

**Theorem 2.1.** Let \((R, m, k)\) be a Gorenstein local ring. Suppose the associated graded ring \( G(R) \) of \( R \) is Cohen-Macaulay. Then the following two numbers are the same:

(a) the index of \( R \), and

(b) the generalized Loewy length of \( R \).

The proof of our main result is based on the following two lemmas which allow us to use reduction argument.
Lemma 2.2. Let \((R, m, k)\) be a local ring and \(s\) an integer. Suppose that \(x \in m/m^2\) is \(R\)-regular and the induced map \(x: m^{i-1}/m^i \rightarrow m^i/m^{i+1}\) is injective for \(1 \leq i \leq s\). Then the \((R/xR)\)-module \(R/(m^s, x)\) is weakly liftable to \(R\).

Proof. By Lemma 1.5, to show that the \((R/xR)\)-module \(R/(m^s, x)\) is weakly liftable to \(R\), it suffices to show that the monomorphism

\[ xR/x(m^s, x) \hookrightarrow (m^s, x)/x(m^s, x) \]

is split. Let \(I = xR \cap m^s\) and \(W = I + m^{s+1}/m^{s+1}\). Since \(m^s/m^{s+1}\) is a vector space over \(k\), we have a direct summand decomposition \(m^s/m^{s+1} = W \oplus V\) as vector spaces. Let \(e_1, \ldots, e_n\) be a basis of \(V\), and let \(y_1, \ldots, y_n\) be a set of preimages of \(e_i\) in \(m^s\). Now let \(B\) be the submodule of \((m^s, x)/x(m^s, x)\) generated by the images of \(y_i\), \(i = 1, \ldots, n\). We claim that

\[ (m^s, x)/x(m^s, x) = xR/x(m^s, x) \oplus B. \]

It is easily seen that the submodules \(A = xR/x(m^s, x)\) and \(B\) generate \((m^s, x)/x(m^s, x)\). Suppose \(z \in A \cap B\). Then \(z = xb = \sum a_iy_i\), where \(b, a_i \in R\). This implies that \(xb \in m^s\). Modulo \(m^{s+1}\), we get \(xb = \sum a_iy_i = 0\) in \(m^s/m^{s+1}\). Therefore, we have \(xb \in m^{s+1}\). By our assumption this means that \(b \in m^s\). Hence, we get \(z = xb = 0\) in \((m^s, x)/x(m^s, x)\). Since \(x\) is \(R\)-regular, we have \(R/(m^s, x) \cong xR/x(m^s, x)\), and the proof is complete.

Using the same argument, we have

Lemma 2.3. Let \((R, m, k)\) be a local ring and \(s\) an integer. Suppose that \(x \in m/m^2\) is \(R\)-regular and the induced map \(x: m^{i-1}/m^i \rightarrow m^i/m^{i+1}\) is injective for \(1 \leq i \leq s\). Then the \(R\)-module \(R/m^s\) is isomorphic to a direct summand of the \(R\)-module \((m^s, x)/xm^s\). In particular, \(R/m^s\) is an epimorphic image of \((m^s, x)\).

Proof. Let \(e_i, y_i, i = 1, \ldots, n\), be as in Lemma 2.2. Let \(D\) be the submodule of \((m^s, x)/xm^s\) generated by the images of \(y_i\), \(i = 1, \ldots, n\). We claim that there is a direct summand decomposition

\[ (m^s, x)/xm^s = xR/xm^s \oplus D. \]

It is easy to check that the modules \(C = xR/xm^s\) is a submodule of \((m^s, x)/xm^s\) and that \(C\) and \(D\) generate module \((m^s, x)/xm^s\). Now let \(z \in C \cap D\). Then we have \(z = xb = \sum a_iy_i\) with \(b, a_i \in R\). This implies that \(xb \in m^s\). Modulo \(m^{s+1}\), we get \(xb = \sum a_iy_i = 0\) in \(m^s/m^{s+1}\). This shows that \(xb \in m^{s+1}\). By our assumption we have \(b \in m^s\). Therefore, \(z = xb = 0\) in \((m^s, x)/xm^s\). Since \(x\) is \(R\)-regular, we have that \(R/m^s \cong xR/xm^s\), and this completes the proof.

Now we are ready to prove our main result.

Proof of Theorem 2.1. It is known that the index of \(R\) is always bounded by the generalized Loewy length of \(R\) [4]. Thus it is sufficient to show that \(\delta_R(R/m^s) \neq 0\) implies that there exists a sop \(x\) of \(R\) such that \(m^s \subseteq (x)\).

Now suppose that \(\delta_R(R/m^s) = 1\). We may assume that \(k\) is an infinite field. Since \(G(R)\) is Cohen-Macaulay, there exists an \(R\)-sequence \(x = x_1, \ldots, x_d\), where \(d = \dim R\), such that \(x_i \in m/m^{2}\) and \(\overline{x_1}, \ldots, \overline{x_d}\) is a regular sequence of \(G(R)\). Now we proceed by induction on \(d\). Suppose \(d = 1\). By Lemma 2.3, we know that \(R/m^s\) is an epimorphic image of the module \((m^s, x_1)\). Therefore,
\( \delta_R((m^s, x_1)) \neq 0 \) by Lemma 1.1. Lemmas 2.2 and 1.5 show that there is an 
\( R \)-module decomposition

\[
(m^s, x_1)/x_1(m^s, x_1) \cong R/(m^s, x_1) \oplus (m^s, x_1)/x_1R.
\]

By Lemma 1.3 we know that \( \delta_R((m^s, x_1)/x_1R) = 0 \), where \( \overline{R} = R/x_1R \). Therefore, we have \( \delta_R(R/(m^s, x_1)) = \delta_R((m^s, x_1)/x_1(m^s, x_1)) = \delta_R((m^s, x_1)) \neq 0 \) by 
Lemmas 1.2 and 1.1. Since \( \dim \overline{R} = 0 \), Lemma 1.4 implies that \( \delta_R(R/(m^s, x_1)) \neq 0 \) if and only if \( m^s \subset (x_1) \).

Assume that \( d > 1 \). Set \( \overline{R} = R/x_1R \). Then the above argument shows that 
\( \delta_R(R/m^s) = 1 \) implies that \( \delta_R(R/(m^s, x_1)) \neq 0 \). Let \( m_1 = m/x_1R \). Then \( m_1 \) is 
the maximal ideal of \( \overline{R} \) and \( R/(m^s, x_1) \cong \overline{R}/m_1^2 \). Also we have \( G(R/x_1R) \cong 
G(R)/x_iG(R) \). Therefore, by inductive hypothesis we get that \( \delta_R(R/m^s) = 1 \) implies that 
\( \delta_R(R/(m^s, x_1, \ldots, x_d)) = 1 \), where \( R_d \) denotes \( R/(x) \). Hence, we get 
\( m^s \subset (x) \) since \( \dim R_d = 0 \). This completes the proof.

**Remark.** There are examples of Gorenstein local rings whose associated graded 
ring are not Cohen-Macaulay, but the conjecture holds. I know no example 
where the conjecture fails.

Now we give some applications.

**Corollary 2.4** [4]. The conjecture holds for hypersurface rings.

**Proof.** Let \( R = S/(f) \) where \( S \) is a complete regular local ring. Since \( G(S) \) is a polynomial ring over a field and \( G(R) = G(S)/fG(S) \), we have that \( G(R) \) is Cohen-Macaulay.

Let \( k \) be a field. A graded \( k \)-algebra \( R = \bigoplus_{i \geq 0} R_i \) is called **homogeneous** 
if \( R_0 = k \) and \( R = k[R_1] \). The homogeneous \( k \)-algebra has a unique graded 
maximum ideal, namely, \( m = \bigoplus_{i \geq 1} R_i \). All definitions we made and the results 
we obtained so far can be transferred accordingly to homogeneous Gorenstein 
algebras. We then have

**Corollary 2.5** [5]. The conjecture holds for homogeneous Gorenstein \( k \)-algebras.

**Proof.** It follows from the fact that \( G(R) = R \).

A quotient ring \( R = S/I \) of a regular local ring \( S \) is called a **strict complete 
intersection** if the associated graded ring \( G(R) \) is a complete intersection. Strict 
complete intersections are complete intersections. Thus we have

**Corollary 2.6.** The conjecture holds for strict complete intersections.

We denote by \( e(R) \) the multiplicity of \( R \) and \( \mu(m) \) the minimal number of 
generators of \( m \). It is true that \( \mu(m) \leq e(R) + \dim(R) - 1 \). When the equality 
holds, \( R \) is said to have minimal multiplicity. We have the following results.

**Corollary 2.7.** Let \( R \) be a Gorenstein local ring of multiplicity at most 4, or with 
\( \mu(m) = e(R) + \dim(R) - 2 \). Then the conjecture holds for \( R \).

**Proof.** Sally showed in [6] that \( G(R) \) is Gorenstein under the assumption.

Recall that, for a Gorenstein local ring \( R \), \( \text{index}(R) = 1 \) if and only if \( R \) is 
a regular local ring [3]. In the case where \( \text{index}(R) = 2 \) we get
Corollary 2.8. Suppose $G(R)$ is Cohen-Macaulay. Then $\text{index}(R) = 2$ if and only if $R$ has minimal multiplicity.

Proof. Here we use the result in [6] that $R$ has minimal multiplicity if and only if there exists a sop $x$ of $R$ such that $m^2 = (x)m$. Our result shows that, under the given condition, this is equivalent to $\delta_R(R/m^2) = 1$.

References


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