LINKS WITH UNLINKING NUMBER ONE ARE PRIME

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(Communicated by Ronald Stern)

ABSTRACT. We prove that a link with unlinking number one is prime.

1. INTRODUCTION

Let $L$ be a link in $S^3$. The *unlinking number of* $L$, $u(L)$, is defined to be the least number of times $L$ must pass through itself in order to transform it into the unlink. It is not hard to see that this is the same as the minimum, over all diagrams of $L$, of the least number of crossing changes in that diagram needed to transform $L$ to the unlink.

Recall that a *connected sum* of links $L_1$ and $L_2$ is any link $L$ obtained by removing the interior of a trivial ball-pair $(B^3_i, B^1_i) \subset (S^3, L_i), \ i = 1, 2$, and then gluing the resulting pairs along their boundaries $(S^2_i, S^0_i)$. It is convenient to write $L = L_1 \# L_2$, although in general $L$ is not uniquely determined by $L_1$ and $L_2$.

A natural conjecture, which is certainly classical at least in the case of knots, is the following.

**Conjecture.** $u(L_1 \# L_2) = u(L_1) + u(L_2)$.

We shall say that $L$ is *prime* if $L = L_1 \# L_2$ implies that $L_1$ or $L_2$ is an unlink. (Note that although this is the definition of primality that is appropriate in our present context, it may not be the usual one; it differs from that given in [KT], for example.) Since $u(L) = 0$ if and only if $L$ is an unlink, the above Conjecture implies that if $u(L) = 1$ then $L$ is prime. In the case of knots, this was proved by Scharlemann [S]. Here we prove the analog for links with more than one component.

**Theorem.** Let $L$ be a link with more than one component such that $u(L) = 1$. Then $L$ is prime.

Received by the editors July 23, 1992.

1991 Mathematics Subject Classification. Primary 57M25.

Key words and phrases. Unlinking number, prime.

The first author was partially supported by NSF grant NSF-DMS-9001478. The second author was partially supported by NSF grants NSF-DMS-8903599, NSF-DMS-9158090 and a Sloan Foundation Fellowship.
An alternative proof of Scharlemann's theorem was given by Zhang [Z], by applying the main result of [GL1] to the 2-fold branched cover of the knot. The present proof follows the same philosophy, using [GL2] instead of [GL1].

2. Proof

Let $K$ be a knot in the interior of an orientable 3-manifold $M$. Let $N(K)$ be a tubular neighborhood of $K$, and let $\alpha$ be the isotopy class of an essential simple closed curve in $\partial N(K)$. The manifold obtained by $\alpha$-Dehn surgery on $K$, which we will denote by $K(\alpha)$, is the result of attaching a solid torus $V$ to $M - \text{int} N(K)$ by identifying $\partial V$ with $\partial N(K)$ in such a way that $\alpha$ bounds a disk in $V$. If $\alpha$ and $\beta$ are two such isotopy classes in $\partial N(K)$ then $\Delta(\alpha, \beta)$ denotes their minimal geometric intersection number.

If $L$ is a link in $S^3$, let $M(L)$ denote the 2-fold branched cover of $L$, with branched covering projection $p : M(L) \to S^3$ and canonical involution $h : M(L) \to M(L)$.

**Lemma 1.** Let $L, L'$ be links in $S^3$ such that $L'$ is obtained from $L$ by a single crossing change. Then there exists a knot $K$ in $M(L)$ such that

(i) $K$ has an $h$-invariant tubular neighborhood $N(K)$ such that $p(N(K))$ is a 3-ball that meets $L$ in an unknotted pair of arcs;

(ii) $M(L')$ is homeomorphic to $K(\alpha)$ for some $\alpha$ with $\Delta(\alpha, \mu) = 2$, where $\mu$ is the meridian of $K$.

**Proof.** This follows from [L, Proof of Lemma 1]. (See also [M].) □

Note that if $L$ is the $n$-component unlink, $n \geq 1$, then

$$M(L) \cong \#_{n-1} S^1 \times S^2.$$ 

The next lemma gives the converse.

**Lemma 2.** Let $L$ be a link in $S^3$ such that $M(L)$ is homeomorphic to $\#_{n-1} S^1 \times S^2$. Then $L$ is the $n$-component unlink.

**Proof.** The case $n = 1$ is the $\mathbb{Z}_2$-Smith Conjecture [W]. The general case follows easily by induction on $n$ using [KT, Lemma 1]. □

Finally, we shall need the following fact about Dehn surgeries that yield reducible manifolds.

**Lemma 3.** Let $K$ be a knot in a 3-manifold $M$ such that $M - K$ is irreducible but $K(\alpha)$ and $K(\beta)$ are reducible. Then

$$\Delta(\alpha, \beta) \leq 1.$$ 

**Proof.** This is [GL2, Corollary 1.2]. □

**Proof of Theorem.** Let $L$ be an $n$-component link in $S^3$, $n \geq 2$, with $u(L) = 1$. First note that if $L$ is a split union $L_1 \sqcup L_2$, then clearly it must be that $L_1$ (say) is an unlink and $u(L_2) = 1$. Hence we may assume that $L$ is nonsplit and, therefore, that $S^3 - L$ is irreducible.
Suppose for contradiction that $L$ is a connected sum $L_1 \# L_2$, where $L_i$ is not an unlink, $i = 1, 2$. Then $M(L) \cong M(L_1) \# M(L_2)$. In particular, since $L_i$ is not the unknot, $i = 1, 2$, $M(L)$ is reducible, by the $\mathbb{Z}_2$-Smith Conjecture [W].

Since $u(L) = 1$, we can apply Lemma 1 with $L'$ the $n$-component unlink, giving a knot $K$ in $M(L)$, with meridian $\mu$, such that

$$K(\alpha) \cong M(L') \cong \#_{n-1} S^1 \times S^2,$$

where $\Delta(\alpha, \mu) = 2$. But $K(\mu) \cong M(L)$ is also reducible. It follows from Lemma 3 that $X = M(L) - \text{int} N(K)$ is reducible.

Choosing $N(K)$ as in part (i) of Lemma 1, let $B_0$ be the 3-ball $S^3 - \text{int} p(N(K))$, and let $L_0 = L \cap B_0$. Then $X$ is the 2-fold branched cover of $(B_0, L_0)$. Since $h$ restricts to an involution on $X$, [KT, Lemma 1] implies that $X$ contains an essential 2-sphere $S$ (one that does not bound a 3-ball) such that either $h(S) \cap S = \emptyset$ or $h(S) = S$ and $S$ meets $p^{-1}(L)$ transversely.

In the first case, $p(S)$ is a 2-sphere in $B_0 - L_0$. Since $S^3 - L$ is irreducible by assumption, $p(S)$ bounds a 3-ball $B$ in $B_0 - L_0$. But then $B$ lifts to a 3-ball in $X$ bounded by $S$, contradicting the essentiality of $S$.

In the second case, $S$ must meet $p^{-1}(L)$ in two points. Then $p(S)$ is a 2-sphere in $B_0$ meeting $L_0$ in two points. Let $B_1$ be the 3-ball in $B_0$ bounded by $p(S)$, and let $L_1 = L \cap B_1$. Then in $X$, $S$ bounds $M_1$, the 2-fold branched cover of $(B_1, L_1)$. Since $M_1 \subset X \subset M(L') \cong \#_{n-1} S^1 \times S^2$, $M_1$ is homeomorphic to $\#_{k-1} S^1 \times S^2$ minus the interior of a 3-ball, where $k \leq n$. By Lemma 2 and the assumption that $L$ is nonsplit, it follows that $L_1$ is an unknotted arc in $B_1$, giving $M_1 \cong B^3$ and again contradicting the essentiality of $S$. □

ADDED IN PROOF

After this paper was accepted for publication we learned that the theorem of the title is contained in Theorem 2 of [E-M]. So our paper should be regarded as giving a new proof of Eudave-Muñoz's result, in which the explicit combinatorial arguments of [E-M] are replaced by a reference to the main result of [GL2] (whose proof is based on more complicated combinatorial arguments). In the same way, our approach gives alternative proofs of Theorems 1, 2, and 3 of [E-M]. Eudave-Muñoz was also aware of the fact that the primality of unknotting number one knots follows from [GL1] (see [E-M, p. 775]).

REFERENCES


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