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ON THE HYPOELLIPTICITY OF CONVOLUTION EQUATIONS IN THE ULTRADISTRIBUTION SPACES OF L^q GROWTH

DUŠANKA KOVAČEVIĆ

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ABSTRACT. We consider convolution equations in the ultradistribution spaces $\mathscr{D}'^{(M_p)}_{L^q}$ and $\mathscr{D}'^{\{M_p\}}_{L^q}$, $q \in [1, \infty]$, of Beurling and Roumieu type of L^q growth. Our main aim is to find conditions for convolution operators to be hypoelliptic in $\mathscr{D}'^{(M_p)}_{L^\infty}$ and $\mathscr{D}'^{\{M_p\}}_{L^\infty}$ respectively, in terms of their Fourier transforms.

1. PRELIMINARY

The results of Malgrange, Ehrenpreis, and Hörmander on the solvability and hypoellipticity of convolution equations in Schwartz's spaces stimulated many mathematicians to study such problems in various subspaces of distributions. We cite here only results of Zieleźny [18, 19] and Pahk [13] since they are connected with our results. The research of the convolution equations in the spaces of ultradistributions was started fifteen years ago by Chou [4] and Grudzinski [7] and was continued by Ciorănescu [6], by Braun, Meise, Taylor, Voigt, and their cooperators (see [1, 12] and references therein), and Pilipović [17], who started investigations of hypoelliptic convolution equations in ultradistribution spaces similarly as it was done for distribution spaces by Zieleźny. In this paper we accept that Denjoy-Carleman-Komatsu approach to the theory of ultradistribution and study hypoelliptic convolution equations in the Beurling and Roumieu ultradistribution spaces $\mathscr{D}'^{(M_p)}_{L^q}$ and $\mathscr{D}'^{\{M_p\}}_{L^q}$, $q \in [1, \infty]$, which are generalizations of the space $\mathscr{D}'_{L^q}$. These spaces have been investigated by Pilipović, Ciorănescu, Carmichael, and Pathak (see [14, 5, 2, 3]). An analogous problem but in the distribution spaces $\mathscr{D}'_{L^q}$, $q \in [1, \infty]$, was investigated by Pahk [13]. Some of Pahk's considerations are easily transferred to the problem that we consider, but many problems appear to be specific for ultradistributions and they have been solved.

Let us recall the basic facts about the ultradistribution spaces $\mathscr{D}'^{(M_p)}_{L^q}$ and $\mathscr{D}'^{\{M_p\}}_{L^q}$, $q \in [1, \infty]$, of L^q growth and the spaces $\mathscr{S}'^{(M_p)}$ and $\mathscr{S}'^{\{M_p\}}$ of tempered ultradistributions. First we give a short survey of the notation which will be used in this paper. The sets of nonnegative integers, natural real, and complex numbers are denoted by $\mathbf{N}_0$, $\mathbf{N}$, $\mathbf{R}$, and $\mathbf{C}$. The letter $\mathscr{C}$ (without

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super- or subscript) denotes a positive constant, not necessarily the same at each occurrence. $\mathscr{R}$ is a family of positive sequences which increase to infinity. The norm in L^r, $r \in [1, \infty]$, is denoted by $\|\cdot\|_r$. The Fourier transform of $\varphi \in L^1$ is given by

$$\mathscr{F}\varphi(\xi) = \hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \varphi(x) e^{-i\xi x}\, dx .$$

M_p is a sequence of positive numbers which satisfy the following conditions [8]:

$$\text{(M.0)} \qquad M_0 = 1 ;$$

$$\text{(M.1)} \qquad M_\alpha^2 \le M_{\alpha-1} M_{\alpha+1}, \qquad \alpha \in \mathbf{N};$$

$$\text{(M.2)} \quad M_\alpha \le A H^\alpha \min_{0 \le \beta \le \alpha} M_\beta M_{\alpha-\beta}, \qquad \alpha \in \mathbf{N}, \text{ for some } A > 0,\ H > 0;$$

$$\text{(M.3)} \qquad \sum_{\alpha+1 \le \beta} \frac{M_{\beta-1}}{M_\beta} \le A\alpha \frac{M_\alpha}{M_{\alpha+1}}, \qquad \alpha \in \mathbf{N}, \text{ for some } A > 0,$$

and $m_p = M_p / M_{p-1}$, $p \in \mathbf{N}$. The letters A and H will denote the constants mentioned in (M.2). The so-called associated function for the sequence M_p is

$$M(\rho) = \sup_{p \in \mathbf{N}_0} \log(\rho^p / M_p), \qquad \rho > 0 .$$

The associated function for the sequence $N_p = M_p(\prod_{1 \le k \le p} a_k)$, $a_p \in \mathscr{R}$, is denoted by N_{a_p}.

For the definition and the properties of the ultradistribution space $\mathscr{D}'^{(M_p)}$ (resp. $\mathscr{D}'^{\{M_p\}}$) of Beurling (resp. Roumieu) type we refer to [8]. The common notation for the symbols (M_p) and $\{M_p\}$ will be $*$.

As in [15], for $r \in [1, \infty]$, we put

$$\mathscr{D}_{L^r}^{(M_p)} = \operatorname{proj\,lim}_{h \to 0} \mathscr{D}_{L^r, h}^{M_p}, \qquad \mathscr{D}_{L^r}^{\{M_p\}} = \operatorname{ind\,lim}_{h \to \infty} \mathscr{D}_{L^r, h}^{M_p},$$

where $\mathscr{D}_{L^r, h}^{M_p}$ is the space of functions ϕ from C^∞ for which

$$\text{(1)} \qquad \|\phi\|_{L^r, h} = \sup_{\alpha \in \mathbf{N}_0} \frac{\|\partial^\alpha \phi\|_r}{h^\alpha M_\alpha} < \infty .$$

Since $\mathscr{D}^*$ is dense in $\mathscr{D}_{L^r}^*$, $r \in [1, \infty)$, and the inclusion mapping is continuous, the corresponding strong duals of $\mathscr{D}_{L^r}^*$, $\mathscr{D}_{L^q}'^*$, $q = r/(r-1)$, are subspaces of ultradistribution spaces. We denote by $\dot{\mathscr{B}}^*$ the completion of $\mathscr{D}^*$ in $\mathscr{D}_{L^\infty}^*$. The strong dual of $\dot{\mathscr{B}}^*$ is $\mathscr{D}_{L^1}'^*$. Let

$$\widetilde{\mathscr{D}}_{L^r}^{\{M_p\}} = \operatorname{proj\,lim}_{l_p \in \mathscr{R}} \mathscr{D}_{L^r, l_p}^{M_p},$$

where $\mathscr{D}_{L^r, l_p}^{M_p}$, $l_p \in \mathscr{R}$, is the space of functions ϕ from C^∞ for which

$$\text{(2)} \qquad \|\phi\|_{L^r, l_p} = \sup_{\alpha \in \mathbf{N}_0} \frac{\|\partial^\alpha \phi\|_r}{(\prod_{1 \le \beta \le \alpha} l_\beta) M_\alpha} < \infty ,$$

and let us denote the completion of $\mathscr{D}^{\{M_p\}}$ in $\widetilde{\mathscr{D}}^{\{M_p\}}$ by $\dot{\widetilde{\mathscr{B}}}^{\{M_p\}}$. The strong dual of $\widetilde{\mathscr{D}}_{L^r}^{\{M_p\}}$ is denoted by $\widetilde{\mathscr{D}}_{L^q}'^{\{M_p\}}$, $q = r/(r-1)$, and the strong dual of

$\dot{\widetilde{\mathscr{B}}}^{\{M_p\}}$ is denoted by $\widetilde{\mathscr{D}}'^{\{M_p\}}_{L^1}$. From [15, Lemma 3(i), (ii)] it follows that in the set theoretical sense $\widetilde{\mathscr{D}}^{\{M_p\}}_{L^r} = \mathscr{D}^{\{M_p\}}_{L^r}$, $r \in (1, \infty)$, and $\dot{\widetilde{\mathscr{B}}}^{\{M_p\}} = \dot{\mathscr{B}}^{\{M_p\}}$ and that the inclusion mappings $i\colon \mathscr{D}^{\{M_p\}}_{L^r} \to \widetilde{\mathscr{D}}^{\{M_p\}}_{L^r}$ and $i\colon \dot{\mathscr{B}}^{\{M_p\}} \to \dot{\widetilde{\mathscr{B}}}^{\{M_p\}}$ are continuous. Hence, $\widetilde{\mathscr{D}}'^{*}_{L^q}$ is a topological subspace of $\mathscr{D}'^{*}_{L^q}$

Let $b > 0$ (resp. $b_p \in \mathscr{R}$)

$$P_b(\xi) = \prod_{\alpha \in \mathbf{N}} \left(1 + \frac{\xi^2}{b^2 m_\alpha^2}\right) \quad \left(\text{resp. } P_{b_p}(\xi) = \prod_{\alpha \in \mathbf{N}} \left(1 + \frac{\xi^2}{b_\alpha^2 m_\alpha^2}\right)\right), \qquad \xi \in \mathbf{R}.$$

P_b (resp. P_{b_p}) is an ultrapolynomial of class (M_p) (resp. $\{M_p\}$) (see [8, Proposition 4.5]), which means that $P(\xi) = \sum_{\alpha \in \mathbf{N}_0} a_\alpha \xi^\alpha$, $\xi \in \mathbf{R}$, where the coefficients a_α satisfy the estimate $|a_\alpha| \le \mathscr{C} l^\alpha M_\alpha$, $\alpha \in \mathbf{N}_0$, for some $l > 0$ and $\mathscr{C}$ (resp. for every $l > 0$ and some $\mathscr{C}$). Applying [9, Lemma 3.4] we get that in the Roumieu case it is equivalent to the assertion "there is a sequence $l_p \in \mathscr{R}$ and $\mathscr{C}$ such that $|a_\alpha| \le \mathscr{C}(\prod_{1 \le \beta \le \alpha} l_\beta) M_\alpha$, $\alpha \in \mathbf{N}_0$". We will need the following two estimates

$$\exp M(b|\xi|) \le P_b(\xi) \quad (\text{resp. } \exp N_{b_p}(|\xi|) \le P_{b_p}(\xi)), \qquad \xi \in \mathbf{R},$$

see [8]; there exist $r > 0$ and $\mathscr{C}$ such that

$$\begin{gathered} \left|\left(\frac{1}{P_b(\xi)}\right)^{(\gamma)}\right| \le \mathscr{C} \frac{\gamma!}{r^\gamma} \exp[-M(2b|\xi|)] \\ \left(\text{resp. } \left|\left(\frac{1}{P_{b_p}(\xi)}\right)^{(\gamma)}\right| \le \mathscr{C} \frac{\gamma!}{r^\gamma} \exp[-N_{2b_p}(|\xi|)]\right), \qquad \xi \in \mathbf{R},\ \gamma \in \mathbf{N}_0. \end{gathered} \tag{3}$$

An ultradistribution T is in $\mathscr{D}'^{(M_p)}_{L^q}$ (resp. $\mathscr{D}'^{\{M_p\}}_{L^q}$), $q \in [1, \infty]$, if and only if there are $b > 0$ (resp. $b_p \in \mathscr{R}$) such that

$$f = P_b(D) F_1 + F_2 \quad (\text{resp. } f = P_{b_p}(D) F_1 + F_2), \qquad D = \frac{1}{i} \frac{\partial}{\partial x},$$

where $F_1, F_2 \in L^q$ [15, Theorem 1].

The space $\mathscr{S}'^{*}$ of tempered ultradistributions is defined and investigated in [11]. In the special case when M_p is a Gevrey's sequence space $\mathscr{S}'^{(M_p)}$ is investigated in [16]. The space $\mathscr{S}_2^{M_p, m}$, $m > 0$, is the space of smooth functions φ which satisfy

$$\sigma_{m,2}(\varphi) = \left(\sum_{\alpha, \beta \in \mathbf{N}_0} \int_{\mathbf{R}} \left|\frac{m^{\alpha+\beta}}{M_\alpha M_\beta} (1 + |x|^2)^{\beta/2} \varphi^{(\alpha)}(x)\right|^2 dx\right)^{1/2} < \infty,$$

equipped with the topology induced by the norm $\sigma_{m,2}$. The spaces $\mathscr{S}'^{(M_p)}$ and $\mathscr{S}'^{\{M_p\}}$ of tempered ultradistributions of Beurling and Roumieu type respectively are defined as the strong duals of

$$\mathscr{S}^{(M_p)} = \lim_{m \to \infty} \operatorname{proj} \mathscr{S}_2^{M_p, m} \quad \text{and} \quad \mathscr{S}^{(M_p)} = \lim_{m \to 0} \operatorname{ind} \mathscr{S}_2^{M_p, m}$$

respectively.

The Fourier transform is an isomorphism of spaces $\mathscr{S}^*$ and $\mathscr{S}'^*$ onto themselves.

The space $\mathscr{O}_M^{(M_p)}$ (resp. $\mathscr{O}_M^{\{M_p\}}$) of multipliers of $\mathscr{S}'^{(M_p)}$ (resp. $\mathscr{S}'^{\{M_p\}}$) consists of ultradifferentiable functions φ such that for every $r > 0$ there is $m > 0$ (resp. for some $r > 0$ and every $m > 0$)

$$\sup_{\alpha \in \mathbf{N}_0} \frac{1}{r^\alpha M_\alpha} \|\exp[-M(m|\cdot|)]\varphi^{(\alpha)}\|_\infty < \infty .$$

In the Beurling case this result is proved in [10], but the proof in the Roumieu case is quite analogous. The Fourier transform is an isomorphism of $\mathscr{O}_M^*$ onto $\mathscr{O}_c'^*$, where $\mathscr{O}_c'^*$ denotes the space of convolution operators-convolutors of $\mathscr{S}'^*$, whose explicit characterization is given in [15, Proposition 9].

2. Hypoellipticity in $\mathscr{D}_{L^q}'^*$, $q \in [1, \infty]$

We define hypoelliptic convolution operators in $\mathscr{D}_{L^\infty}'^*$ as follows: *An ultradistribution $S \in \mathscr{D}_{L^1}'^*$ is hypoelliptic in $\mathscr{D}_{L^\infty}'^*$ if every solution U in $\mathscr{D}_{L^\infty}'^*$ of the convolution equation*

$$S * U = V \tag{4}$$

belongs to $\mathscr{D}_{L^\infty}^*$, when V is in $\mathscr{D}_{L^\infty}^*$. In that case equation (4) is also called hypoelliptic in $\mathscr{D}_{L^\infty}'^*$.

The space of convolution operators in $\mathscr{D}_{L^\infty}'^*$ is $\mathscr{D}_{L^1}'^*$; therefore, hypoelliptic convolution operators in $\mathscr{D}_{L^\infty}'^*$ have to be characterized as a subspace of $\mathscr{D}_{L^1}'^*$. Because of lack of differentiability of their Fourier transforms, in this paper we consider only the subclasses of $\mathscr{D}_{L^1}'^*$ containing $\mathscr{O}_c'^*$, whose Fourier transforms are C^∞-functions of ultrapolynomial growth. In this class we characterize hypoelliptic convolution operators in $\mathscr{D}_{L^\infty}'^*$. But we have an example of a hypoelliptic convolution operator in $\mathscr{D}_{L^\infty}'^*$ which is not in this class.

We will now establish a necessary and sufficient condition for a convolution operator to be hypoelliptic in $\mathscr{D}_{L^\infty}'^*$. The result is proved only for subclasses of convolution operators in $\mathscr{D}_{L^\infty}'^*$, and the proof is based on an idea similar to the one for distribution spaces used in [13, 18, 19]. However, some problems appear to be specific for ultradistributions, and they have been solved.

Definition 1. $S \in \mathscr{D}_{L^1}'^{(M_p)}$ (resp. $\mathscr{D}_{L^1}'^{\{M_p\}}$) is said to be of class H_a, $a > 0$ (resp. H_{a_p}, $a_p \in \mathscr{R}$), if the Fourier transform $\widehat{S}$ is a C^∞-function such that there exists $l > 0$ (resp. $(l_p) \in \mathscr{R}$) for which

$$\begin{gathered} \sum_{\alpha \in \mathbf{N}} \frac{1}{a^\alpha M_\alpha} \widehat{S}^{(\alpha)}(x) = \mathscr{O}(\exp[M(l|x|)] \\ \left(\text{resp. } \sum_{\alpha \in \mathbf{N}} \frac{1}{(\prod_{1 \le \beta \le \alpha} a_\beta) M_\alpha} \widehat{S}^{(\alpha)} = \mathscr{O}(\exp[N_{l_p}(|x|)]\right), \qquad |x| \to \infty . \end{gathered} \tag{5}$$

The above-defined class of ultradistributions will be used for our study of hypoellipticity in $\mathscr{D}_{L^\infty}'^{(M_p)}$ (resp. $\mathscr{D}_{L^\infty}'^{\{M_p\}}$).

Lemma 1. *Let S be an ultradistribution whose Fourier transform is of the form*

$$\widehat{S} = \sum_{j \in \mathbf{N}} a_j \delta_{\xi_j} , \tag{6}$$

where ξ_j *is a sequence of real numbers such that*

$$2^j < 2|\xi_{j-1}| < |\xi_j|, \qquad j \in \mathbf{N}, \tag{7}$$

and a_j *are complex numbers such that for some (resp. each)* $m > 0$

$$|a_j| = \mathscr{O}(\exp[M(m|\xi_j|)]). \tag{8}$$

(1) S *belongs to* $\mathscr{D}'^{(M_p)}_{L^\infty}$ *(resp.* $\mathscr{D}'^{\{M_p\}}_{L^\infty}$*).*

(2) S *belongs to* $\mathscr{D}^{\{M_p\}}_{L^\infty}$ *(resp.* $\mathscr{D}^{(M_p)}_{L^\infty}$*) if and only if for each (resp. some)* $k > 0$

$$|a_j| = o(\exp[-M(k|\xi_j|)]). \tag{9}$$

Note that according to [9, Lemma 3.4] in the Roumieu case condition (8) (i.e., "for each $m > 0$ (8) holds") is equivalent to

$$|a_j| = \mathscr{O}(\exp[N_{m_p}(|\xi_j|)]) \tag{10}$$

for some $m_p \in \mathscr{R}$, and condition (9) is equivalent to

$$|a_j| = o(\exp[-N_{k_p}(|\xi_j|)]) \tag{11}$$

for each $k_p \in \mathscr{R}$.

Proof. (1) Let us first prove that the sum $S = \sum_{j\in\mathbf{N}} a_j e^{ix\xi_j}$ converges in $\mathscr{D}'^{(M_p)}_{L^\infty}$ (resp. $\mathscr{D}'^{\{M_p\}}_{L^\infty}$). Suppose that $\varphi \in \mathscr{D}^{(M_p)}_{L^1}$ (resp. $\mathscr{D}^{\{M_p\}}_{L^1}$). Using the fact that for each ultrapolynomial P of class (M_p) (resp. $\{M_p\}$) satisfies

$$|P(\xi)\mathscr{F}^{-1}\varphi(\xi)| \le \|P(D)\varphi\|_1,$$

we conclude that, for each $b > 0$, some $l > 0$ and $\mathscr{C}$ (resp. for each $b_p \in \mathscr{R}$, some $l_p \in \mathscr{R}$ and $\mathscr{C}$)

$$\begin{gathered} |\mathscr{F}^{-1}\varphi(\xi)| \le \frac{\mathscr{C}}{P_b(\xi)} \sum_{\alpha\in\mathbf{N}_0} \frac{1}{l^\alpha M_\alpha} \|\varphi^{(\alpha)}\|_1 \\ \left(\text{resp. } |\mathscr{F}^{-1}\varphi(\xi)| \le \frac{\mathscr{C}}{P_{b_p}(\xi)} \sum_{\alpha\in\mathbf{N}_0} \frac{1}{(\prod_{1\le\beta\le\alpha} l_\beta) M_\alpha} \|\varphi^{(\alpha)}\|_1\right), \qquad \xi \in \mathbf{R}. \end{gathered} \tag{12}$$

Hence, for $b > 0$, such that $b < 2m$ and some $l > 0$ (resp. $b_p \in \mathscr{R}$, such that $b_p < 2m_p$, $p \in \mathbf{N}$, and some $l_p \in \mathscr{R}$)

$$\begin{aligned} |\langle S, \varphi\rangle| &\le \sum_{j\in\mathbf{N}} |a_j \langle e^{ix\xi_j}, \varphi\rangle| \le \sum_{j\in\mathbf{N}} |a_j| |\mathscr{F}^{-1}\varphi(\xi_j)| \\ &\le \mathscr{C} \sum_{j\in\mathbf{N}} \exp[M(m|\xi_j|) - M(2b|\xi_j|)] \left(\sum_{\alpha\in\mathbf{N}} \frac{1}{l^\alpha M_\alpha} \|\varphi^{(\alpha)}\|_1\right) \\ &\le \mathscr{C} \left(\sum_{\alpha\in\mathbf{N}_0} \frac{l^\alpha}{M_\alpha} \|\varphi^{(\alpha)}\|_1\right) \\ &\left(\text{resp. } |\langle S, \varphi\rangle| \le \mathscr{C} \left(\sum_{\alpha\in\mathbf{N}_0} \frac{1}{(\prod_{1\le\beta\le\alpha} l_\beta) M_\alpha} \|\varphi^{(\alpha)}\|_1\right)\right), \end{aligned}$$

which imply that S converges in $\mathscr{D}'^{(M_p)}_{L^\infty}$ (resp. $\widetilde{\mathscr{D}}'^{\{\infty\}}_{L^\infty}$). In the Roumieu case the convergence in $\widetilde{\mathscr{D}}'^{\{M_p\}}_{L^\infty}$ implies that S converges in $\mathscr{D}'^{\{M_p\}}_{L^\infty}$.

(2) Suppose that S belongs to $\mathscr{D}^{(M_p)}_{L^\infty}$ (resp. $\mathscr{D}^{\{M_p\}}_{L^\infty}$). For every ultradifferential operator $P(D)$ of class (M_p) (resp. $\{M_p\}$) and every $\varphi \in \mathscr{D}^{(M_p)}_{L^1}$ (resp. $\mathscr{D}^{\{M_p\}}_{L^1}$)

$$\langle e^{ixu}(P(D)S(x)), \varphi(x)\rangle \to 0 \quad \text{as } |u| \to \infty, \ u \in \mathbf{R}.$$

This follows from the next calculation

$$\begin{aligned}|\langle e^{ixu}(P(D)S(x)), \varphi(x)\rangle| &= \left|\frac{1}{u}\int_{\mathbf{R}} (P(D)S(x))\varphi(x) De^{ixu}\,dx\right| \\ &\le \frac{1}{|u|}\int_{\mathbf{R}} |D(P(D)S(x))\varphi(x)|\,dx \le \frac{\mathscr{C}}{|u|}.\end{aligned}$$

Note, $\mathscr{D}^*_{L^1} \subset \mathscr{D}^*_{L^2}$, $\varphi \in L^2$. Passing to the Fourier transform we get

$$\begin{aligned}\langle e^{ixu}(P(D)S(x)), \varphi(x)\rangle &= \langle \mathscr{F}(e^{ixu}(P(D)S(x)))(\xi), \mathscr{F}(\varphi)(\xi)\rangle \\ &= \langle P(\xi+u)\widehat{S}(\xi+u), \hat{\varphi}(\xi)\rangle \\ &= \langle \widehat{S}(\xi), P(\xi)\hat{\varphi}(\xi-u)\rangle.\end{aligned}$$

Therefore, for each ultradifferential operator $P(D)$ of class (M_p) (resp. $\{M_p\}$)

$$\sum_{j\in\mathbf{N}} a_j P(\xi_j)\hat{\varphi}(\xi_j - u) \to 0 \quad \text{as } |u| \to \infty, \ u \in \mathbf{R}. \tag{13}$$

Let us fix $\varphi \in \mathscr{D}'^{(M_p)}_{L^1}$ (resp. $\mathscr{D}'^{\{M_p\}}_{L^1}$) so that

$$|\hat{\varphi}(0)| \ge 1 \tag{14}$$

and

$$\hat{\varphi}(\xi) = 0 \quad \text{for } |\xi| \ge 1. \tag{15}$$

Suppose that condition (9) is not satisfied. There is $c \in \mathbf{N}$ (resp. $c_p \in \mathscr{R}$) and $A > 0$ such that

$$\exp[M(c|\xi_j|)]|a_j| \ge A \quad (\text{resp. } \exp[N_{c_p}(|\xi_j|)]|a_j| \ge A) \tag{16}$$

for a subsequence of a_j, which we may take as the whole sequence without loss of generality. Let $u_j = \xi_j$, $j \in \mathbf{N}$. Making use of (7) and (15) we obtain

$$\sum_{\substack{j\in\mathbf{N}\\ j\ne k}} a_j P(\xi_j)\hat{\varphi}(\xi_j - u_k) = 0.$$

On the other hand, conditions (14) and (16) imply that if $P = P_c$ (resp. $P = P_{c_p}$)

$$|a_k|P_c(\xi_k)\hat{\varphi}(0) \ge A \quad (\text{resp. } |a_k|P_{c_p}(\xi_k)\hat{\varphi}(0) \ge A).$$

This contradicts the convergence (13).

Conversely, if (9) holds then

$$\sup_{\alpha\in\mathbf{N}_0} \frac{h^\alpha}{M_\alpha}\|S^{(\alpha)}\|_\infty \le \sup_{\alpha\in\mathbf{N}_0}\sum_{j\in\mathbf{N}} \|a_j(i\xi_j)^\alpha e^{i\xi_j}\|_\infty < \infty,$$

which implies $S \in \mathscr{D}^{(M_p)}_{L^\infty}$ (resp. $S \in \mathscr{D}^{\{M_p\}}_{L^\infty}$). $\square$

Theorem 1. *Let S be an ultradistribution in $\mathscr{D}'^{(M_p)}_{L^1}$ (resp. $\mathscr{D}'^{\{M_p\}}_{L^1}$) which is of class H_a (resp. H_{a_p}). Then S is hypoelliptic in $\mathscr{D}'^{(M_p)}_{L^\infty}$ (resp. $\mathscr{D}'^{\{M_p\}}_{L^\infty}$) if and only if there exist $k > 0$ and $\xi_0 > 0$ (resp. for every $k > 0$ there exists $\xi_0 > 0$) such that*

$$|\widehat{S}(\xi)| \geq \exp[M(k|\xi|)], \qquad \xi \in \mathbf{R},\ |\xi| \geq \xi_0. \tag{17}$$

Proof. (1) Suppose that condition (17) is not fulfilled. Then there exists a sequence ξ_j defined as in Lemma 1 and such that

$$|\widehat{S}(\xi_j)| < \exp[M(j|\xi_j|)], \qquad j \in \mathbf{N}.$$

The series $U = \sum_{j\in\mathbf{N}} e^{ix\xi_j}$ converges in $\mathscr{D}'^{*}_{L^\infty}$, but it does not in $\mathscr{D}^{*}_{L^\infty}$. On the other hand,

$$S * U = \sum_{j\in\mathbf{N}} \widehat{S}(\xi_j) e^{ix\xi_j}.$$

Applying Lemma 1 we conclude that $S * U$ is in $\mathscr{D}^{*}_{L^\infty}$. Thus S is not hypoelliptic in $\mathscr{D}'^{*}_{L^\infty}$.

(2) Let ψ be an element of $\mathscr{D}^{(M_p)}$ (resp. $\mathscr{D}^{\{M_p\}}$), such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| < |\xi_0|, \\ 0, & |\xi| \geq |\xi_0| + 1. \end{cases}$$

We define the Fourier transform $\widehat{R}$ of R by the formula

$$\widehat{R}(\xi) = \begin{cases} 0, & |\xi| < |\xi_0|, \\ (1 - \psi(\xi))/\widehat{S}(\xi), & |\xi| \geq |\xi_0|. \end{cases}$$

The above definition of R has sense since S satisfies (17).

There exists $b > 0$ (resp. $b_p \in \mathscr{R}$) such that

$$\widehat{Q}(\xi) = \widehat{R}(\xi)/P_b(\xi) \quad (\text{resp. } \widehat{Q}(\xi) = \widehat{R}(\xi)/P_{b_p}(\xi)),$$

and all its derivatives belong to L^1. We will prove it only in the Roumieu case, since the proof in the Beurling case is analogous. By the iterated "chain rule"

$$\partial^\alpha\left(\frac{1}{\widehat{S}}\right) = \sum_{1\leq\gamma\leq\alpha} \sum_{\alpha_1+\cdots+\alpha_\gamma=\alpha} \mathscr{C}_{\alpha_1\alpha_2\cdots\alpha_\gamma} \frac{(\partial^{\alpha_1}\widehat{S})(\partial^{\alpha_2}\widehat{S})\cdots(\partial^{\alpha_\gamma}\widehat{S})}{\widehat{S}^{\gamma+1}}, \qquad \alpha \in \mathbf{N}.$$

Applying estimates (3), of derivatives of $1/P_b$ (resp. $1/P_{b_p}$), we obtain that for each fixed $\alpha \in \mathbf{N}_0$ and $b_p \in \mathscr{R}$ such that $2b_p > H^{\alpha-1}l_p$, $p \in \mathbf{N}$, where $l_p \in \mathscr{R}$ is the same as in (5), there exist $r > 0$, $\mathscr{C}$, and $\mathscr{C}_\alpha = \mathscr{C}(\alpha)$ such that

$$\begin{aligned}
|\partial^\alpha \widehat{Q}(\xi)| &\leq \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \left|\left(\frac{1}{P_{b_p}(\xi)}\right)^{(\alpha-\beta)}\right| \left|\left(\frac{1}{\widehat{S}}(\xi)\right)^{(\beta)}\right| \\
&\leq \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \frac{(\alpha-\beta)!}{r^{\alpha-\beta}} \exp[-N_{b_p}(|\xi|)] \\
&\quad \cdot \sum_{1\leq\gamma\leq\beta} \sum_{\beta_1+\cdots+\beta_\gamma=\beta} \mathscr{C}^\gamma \mathscr{C}_{\beta_1\beta_2\cdots\beta_\gamma} \frac{m^\beta M_{\beta_1} M_{\beta_2}\cdots M_{\beta_\gamma} \exp[\gamma N_{l_p}(|\xi|)]}{\exp[(\gamma+1)M(k|\xi|)]} \\
&\leq \mathscr{C}_\alpha \exp[-M(k|\xi|)], \qquad \xi \in \mathbf{R}.
\end{aligned}$$

Therefore, the following integration by parts

$$|Q(x)| = \frac{1}{2\pi}\left|\int_{\mathbf{R}} e^{ix\xi}\widehat{Q}(\xi)\,d\xi\right|$$
$$\leq \frac{1}{2\pi}\frac{1}{(1+|x|^2)^{\alpha/2}}\left|\int_{\mathbf{R}} e^{ix\xi}(1-\Delta)^{\alpha/2}\widehat{Q}(\xi)\,d\xi\right| < \mathscr{C}\frac{1}{(1+|x|^2)^{\alpha/2}},$$

where $\mathscr{C}$ depends on the choice of α, has sense. It follows that Q is an L^1 function and so the ultradistribution $R = P_b(D)Q$ (resp. $R = P_{b_p}(D)Q$) is in $\mathscr{D}'^{(M_p)}_{L^1}$ (resp. $\mathscr{D}'^{\{M_p\}}_{L^1}$). Furthermore,

$$\widehat{S}(\xi)\widehat{R}(\xi) = 1 - \psi(\xi).$$

By the inverse Fourier transform, we see that R is a parametrix for S, that is, $S * R = \delta - W$, where $\widehat{W} = \psi$.

Now assume that $S * U = V$, where $V \in \mathscr{D}^*_{L^\infty}$ and $U \in \mathscr{D}'^*_{L^\infty}$. We have

$$U = U * \delta = U * (S * R) + U * W = (U * S) * R + U * W = V * R + U * W.$$

It is easy to check that $V * R$ and $U * W$ belong to $\mathscr{D}^*_{L^\infty}$, and so U is in $\mathscr{D}^*_{L^\infty}$. □

The fact that the Fourier transform is a topological isomorphism from $\mathscr{O}'^*_c$ onto $\mathscr{O}^*_M$ implies that every ultradistribution in $\mathscr{O}'^{(M_p)}_c$ (resp. $\mathscr{O}'^{\{M_p\}}_c$) is of class H_a (resp. H_{a_p}). Therefore,

Corollary 1. *Let S be an ultradistribution in $\mathscr{O}'^{(M_p)}_c$ (resp. $\mathscr{O}'^{\{M_p\}}_c$). Then S is hypoelliptic in $\mathscr{D}'^{(M_p)}_{L^\infty}$ (resp. $\mathscr{D}'^{\{M_p\}}_{L^\infty}$) if and only if there exist $k > 0$ and $\xi_0 > 0$ (resp. for every $k > 0$ there exists $\xi_0 > 0$) such that* (17) *holds.*

Corollary 2. *The same assumptions as in Theorem* 1 *imply that every solution U in $\mathscr{D}'^*_{L^q}$, $q \in [1, \infty]$, of the equation* (4) *is in $\mathscr{D}^*_{L^q}$ whenever V is in $\mathscr{D}^*_{L^q}$.*

Proof. Analogously as in the proof of the sufficiency of the theorem, $R \in \mathscr{D}'^*_{L^1}$ and $U = V * R + U * W$. Since $\mathscr{D}^*_{L^q} * \mathscr{D}'^*_{L^1} \subset \mathscr{D}^*_{L^q}$, we have that U is in $\mathscr{D}^*_{L^q}$. □

If the given convolution operator S is in $\mathscr{D}'^*_{L^1}$ then we have the following weak version of the regularity theorem.

Theorem 2. *If an ultradistribution $S \in \mathscr{D}'^*_{L^1}$ satisfies condition* (17), *then every solution U in $\mathscr{D}'^*_{L^\infty}$ of the equation* (4) *with $V \in \mathscr{D}^*_{L^2}$ is in $\mathscr{D}^*_{L^\infty}$.*

Proof. Applying the same argument as in Theorem 1, we construct the continuous function $\widehat{R}(\xi)$ and $b > 0$ (resp. $b_p \in \mathscr{R}$) so that

$$\widehat{Q}(\xi) = \widehat{R}(\xi)/P_b(\xi) \quad (\text{resp. } \widehat{Q}(\xi) = \widehat{R}(\xi)/P_{b_p}(\xi)), \qquad \xi \in \mathbf{R},$$

is in L^2. By Planchard's theorem Q is in L^2 and $R = P_b(D)Q$ (resp. $R = P_{b_p}(D)Q$) is in $\mathscr{D}'^*_{L^2}$. Also

$$U = U * \delta = V * R + U * V.$$

Since V is in $\mathscr{D}^*_{L^2}$ and $V * R$ and $U * W$ belong to $\mathscr{D}^*_{L^\infty}$, U is in $\mathscr{D}^*_{L^\infty}$. □

We give now two examples of hypoelliptic operators, one of which is not of class H_a (resp. H_{a_p}).

Example 1. Let $S = e^{-|t|}$. Since $\widehat{S}(\xi) = 1/(1+\xi^2)$ is in $\mathscr{O}_c'^*$ and satisfies the condition of the Theorem 1, S is hypoelliptic in $\mathscr{D}_{L^\infty}'^*$.

Example 2. Let $S = 1/(1+x^2)+\delta$. Its Fourier transform $\widehat{S}(\xi) = e^{-|\xi|}+1$ is not a C^1-function, but it satisfies condition (17). From the fact that $1/(1+x^2) \in \mathscr{D}_{L^1}^*$, which follows from the estimation

$$|(1/(1+\xi^2))^{(\beta)}| \leq 3^\beta(\beta+1)!, \qquad \xi \in \mathbf{R},$$

and the fact that $\mathscr{D}_{L^1}^* * \mathscr{D}_{L^\infty}'^* \subset \mathscr{D}_{L^\infty}^*$, it follows that S is a hypoelliptic convolution operator in $\mathscr{D}_{L^\infty}^*$.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF NOVI SAD, TRG DOSITEJA OBRADOVIĆA 4, 21000 NOVI SAD, YUGOSLAVIA

E-mail address: `dusanka%unsim@yubgef51.bitnet`